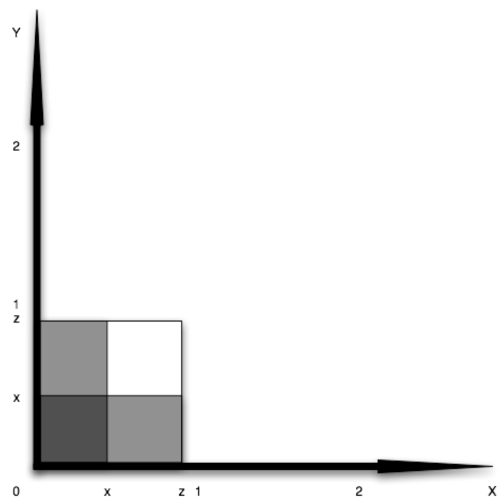
Problem Set #3

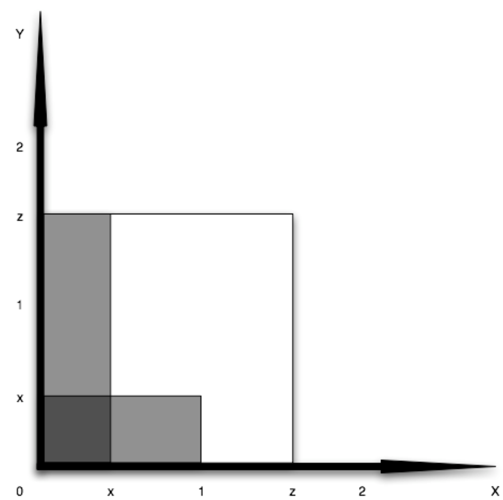
1. ***Max and Min* – 4 pts**

a) **2 pts**

By observation, we know that W ≤ Z.

Two cases: on [0;1] and on [1;2].





On the interval [0;1]:

$$F_{W,Z}(w,z) = \frac{1}{2}\left[zw + w(z-w)\right] = zw - \frac{w^2}{2}$$

$$f_{W,Z}(w,z) = \frac{\partial^2 F_{W,Z}(w,z)}{\partial w \partial z} = 1$$

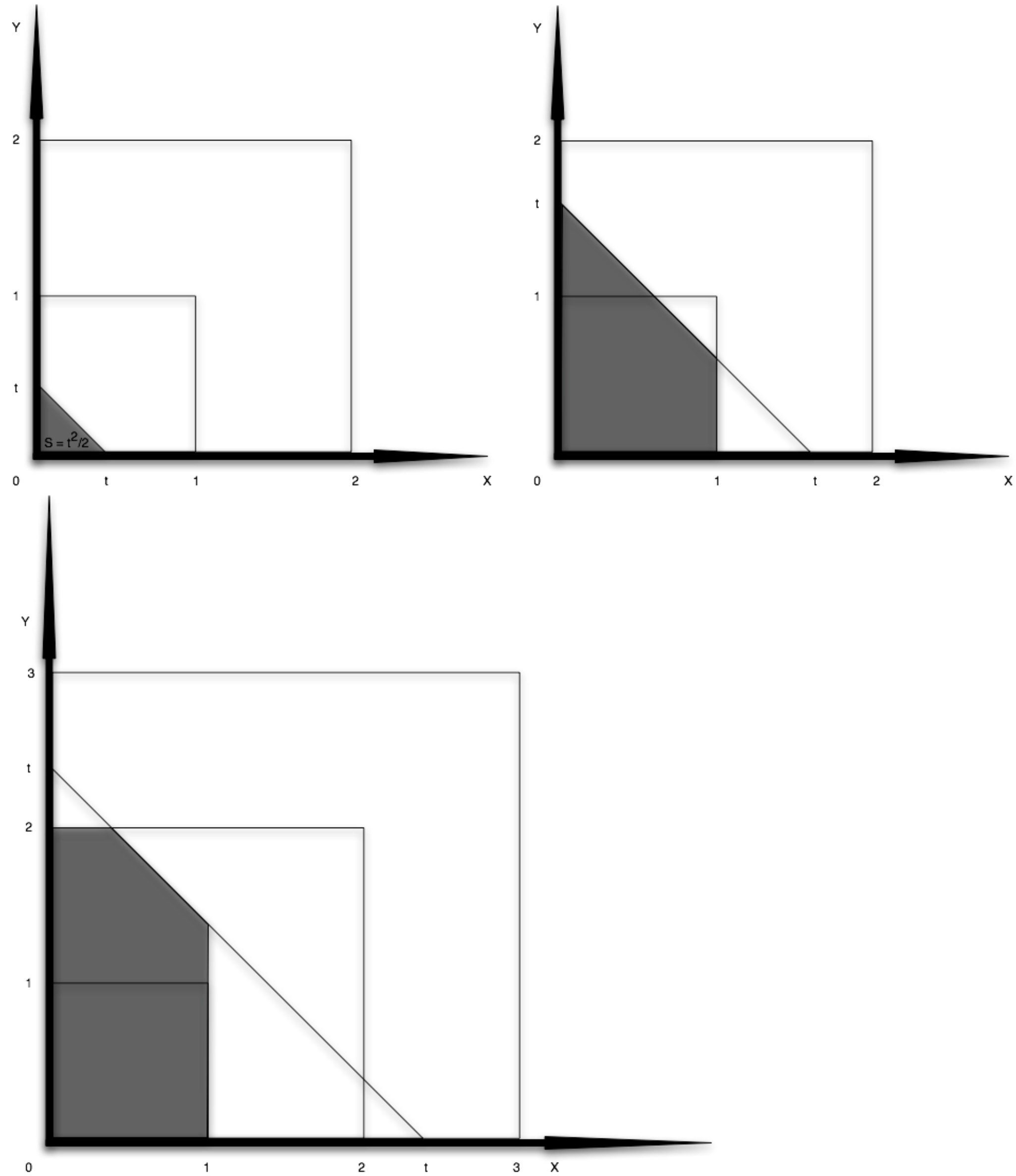
On the interval [1;2]:

$$F_{W,Z}(w,z) = \frac{1}{2}\left[zw + w(1-w)\right] = \frac{w + wz - w^2}{2}$$

$$f_{W,Z}(w,z) = \frac{\partial^2 F_{W,Z}(w,z)}{\partial w \partial z} = \frac{1}{2}$$

b) **2 pts**
By observation, we see that T = W + Z = X + Y. X and Y are independently distributed.

We have three cases: on [0;1], on [1;2] and on [2;3].



On the interval [0;1]:

$$F_T = \frac{1}{2}\times\frac{t^2}{2} = \frac{t^2}{4} \Rightarrow f_T = \frac{t}{2}$$

On the interval [0;1]:

$$F_T = \frac{1}{2}\times 1\times\frac{t+\left[1-(2-t)\right]}{2} = \frac{t}{2}-\frac{1}{4} \Rightarrow f_T = \frac{1}{2}$$

On the interval [0;1]:

$$F_T = \frac{1}{2}\times 1\times\left[2-\frac{(3-t)^2}{2}\right] = 1-\frac{(3-t)^2}{4} \Rightarrow f_T = \frac{3-t}{2}$$

2. ***Building a car* – 5 pts**

a) **2 pts**

Mean time between arrivals of fidgets:

$$E[T_F] = \frac{1}{\lambda} = 10\_\min$$

Mean time between arrivals of whoosies:

$$E[T_W] = \frac{1}{2}\left[\int_0^\infty t\times\left(0.1\times e^{-0.1\times t} + 0.02\times e^{-0.02\times t}\right)dt\right] = 30\_\min$$

b.1) **1 pt**

The Poisson process being memoryless, the mean time from our return to the workstation until the first fidget arrives is 10 min.

b.2) **1 pt**

Mean time to arrival of first whoosy:

$$E[V_W] = E^2[T_W]\frac{1+\left(\dfrac{\sigma_W}{E[T_W]}\right)^2}{2E[T_W]} = 30^2\frac{1+\dfrac{E[T_W{}^2]-E^2[T_W]}{E^2[T_W]}}{60} = 15\times\left[1+\frac{\dfrac{1}{\lambda_1{}^2}+\dfrac{1}{\lambda_2{}^2}-30^2}{30^2}\right]$$

$$E[V_W] = 15\times\left[1+\frac{2500+100-900}{900}\right] = 43.33\_\min$$

b.3) **1pt**

We can have two cases here: either the fidget arrives first, or it's the whoosy.

$$P\{Fidget\_first\} = \int_0^{43.3}\lambda.e^{-\lambda.t}.dt = \int_0^{43.3}0.1\times e^{-0.1\times t} = 0.987$$

If the fidget arrives first, then the total waiting time is the mean waiting time for the first whoosy to arrive: 43.33 min.

If the whoosy is first, however, we will have to wait 10 more minutes on average, since the Poisson process is memoryless.

Therefore: $E\{Total\_waiting\_time\} = 0.987\times 43.33 + (1-0.987)\times(43.33+10) = 43.46\_\min$

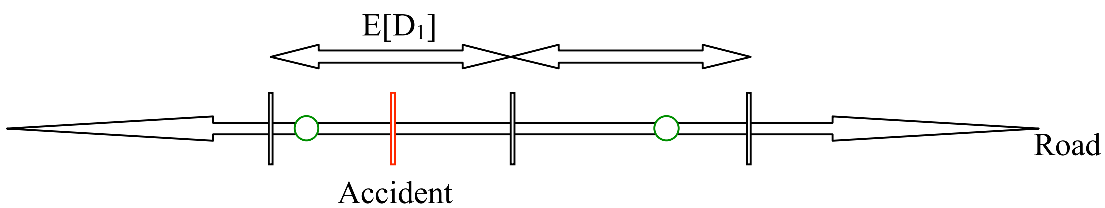
3. *Spatial Poisson on a line* – 4 pts

a) **2 pts**

The distance to first ambulance obeys to a Poisson process with a pdf $d_1(t) = \gamma.e^{-\gamma.t}$

D_1 is the corresponding cumulative distribution function.

Therefore: $E[D_1] = \frac{1}{\gamma}\_mile$ is the average length we have to check before finding an ambulance.



This distance is centered on the accident: the average distance from the accident to the nearest ambulance (E[D'1]) is half this value. Since the ambulance is driven at 1 mile per minute,

$$E[T'_1] = \frac{1}{2.\gamma}\_\min$$

b) **2 pts**

Similarly, we will again have to cover $\frac{1}{\gamma}$ mile on average to find the next ambulance, starting on the border of the area previously searched (memoryless property of the Poisson pdf). However, since we have already determined on which side was the previous ambulance, we know that the second ambulance is necessarily on the other direction.

Therefore, the distance to the second nearest ambulance is: $E[D'_2] = \frac{1}{2.\gamma} + \frac{1}{\gamma} = \frac{3}{2.\gamma}\_miles$ and

$$E[T'_2] = \frac{1}{2.\gamma} + \frac{1}{\gamma} = \frac{3}{2.\gamma}\_\min$$

4. *Covered rectangle* – 4 pts

Coverage rectangle has sides of length L_1 = [Max(X) – Min(X)] and L_2 = [Max(Y) – Min(Y)]. Its area is A = L_1 x L_2.

$$F_M(x) = P\{Max(X) \le x\} = \prod_{i=1}^{10} P[X_i \le x] = \left(\frac{x}{2}\right)^{10}, x \in [0;2]$$

$$f_M(x) = 5\left(\frac{x}{2}\right)^9$$

$$E[Max(X)] = \int_0^2 x \times 5 \times \left(\frac{x}{2}\right)^9 dx = \frac{5}{512}\left[\frac{x^{11}}{11}\right]_0^2 = \frac{20}{11}$$

$$F_m(x) = P\{Min(X) \le x\} = 1 - P\{Min(X) \ge x\} = 1 - \prod_{i=1}^{10} P[X_i \ge x] = 1 - \left(\frac{2-x}{2}\right)^{10}, x \in [0;2]$$

$$f_M(x) = 5\left(\frac{2-x}{2}\right)^9$$

$$E[Min(X)] = \int_0^2 x \times 5 \times \left(\frac{2-x}{2}\right)^9 dx = \frac{5}{512}\left(\left[\frac{x(2-x)^{10}}{-10}\right]_0^2 - \int_0^2 \left(\frac{2-x}{-10}\right)^{10} dx\right) = \frac{5}{512}\left[\frac{(2-x)^{11}}{-110}\right]_0^2 = \frac{2}{11}$$

Therefore, $E[L_1] = \frac{20}{11} - \frac{2}{11} = \frac{18}{11}$

$$F_M(y) = P\{Max(Y) \le y\} = \prod_{i=1}^{10} P[Y_i \le y] = \left(\frac{y}{10}\right)^{10}, y \in [0;10]$$

Similarly, $f_M(y) = \left(\frac{y}{2}\right)^9$

$$E[Max(Y)] = \int_0^{10} y \times \left(\frac{y}{10}\right)^9 dy = \frac{1}{10^9}\left[\frac{y^{11}}{11}\right]_0^{10} = \frac{100}{11}$$

$$F_m(y) = P\{Min(Y) \le y\} = 1 - P\{Min(Y) \ge y\} = 1 - \prod_{i=1}^{10} P[Y_i \ge y] = 1 - \left(\frac{10-y}{2}\right)^{10}, y \in [0;10]$$

$$f_M(x) = \left(\frac{10-y}{10}\right)^9$$

$$E[Min(X)] = \int_0^{10} y \times \left(\frac{10-y}{10}\right)^9 dy = \frac{1}{10^9}\left(\left[\frac{y(10-y)^{10}}{-10}\right]_0^{10} - \int_0^{10} \left(\frac{10-y}{-10}\right)^{10} dy\right) = \frac{1}{10^9}\left[\frac{(10-y)^{11}}{110}\right]_0^{10} = \frac{10}{11}$$

Therefore, $E[L_2] = \frac{100}{11} - \frac{10}{11} = \frac{90}{11}$

As a consequence, $E[A] = E[L_1] \times E[L_2] = \frac{18}{11} \times \frac{90}{11} = \frac{1620}{121} \approx 13.39\_miles^2 \approx 33.67\_km^2$

5. *Cookies, etc* – 3pts

No standardized answer for this question, obviously.