1. Twenty-Six Doors

Let's have: P(A) = Probability that the dollar bill is in envelope A, and E = Evidence (opening of all envelopes except the envelope chosen and envelope Q).

Then, $P(L/E) = \dfrac{P(E/L) \times P(L)}{\displaystyle\sum_{i=A}^{Z} P(E/Letter_i) \times P(Letter_i)} = \dfrac{P(E/L)}{P(E/L) + P(E/Q)} = \dfrac{P(E/L)}{P(E/L) + 1}$

If we consider that the professor would choose the envelope Q at random in the case where the dollar bill is in envelope L, then P(E/L)=1/25.

Consequently, $P(L/E) = \dfrac{\dfrac{1}{25}}{\dfrac{1}{25} + 1} = \dfrac{1}{26}$

Then, $P(Q/E) = \overline{P(L/E)} = 1 - P(L/E) = \dfrac{25}{26}$

The student should choose envelope Q.

If the professor consistently chooses envelope Q when the dollar bill is in envelope L, then P(E/L)=1, and there is an equal chance than the dollar bill be in one envelope or the other.

2. Weird Country

a)
Let's have S = Average *number of children*:

$$S = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

b)
Since there is exactly one daughter in each family, the proportion of daughters among the completed family children is 50%.

c)
Now, we have

$$S = \frac{1}{2} \times 1 + \frac{1}{2^2} \times 2 + \frac{1}{2^3} \times 3 + \frac{1}{2^4} \times 4 + \left(1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4}\right) \times 5 = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{16} = \frac{31}{16}$$

$$DaughterNb = \frac{1}{2} \times 1 + \frac{1}{2^2} \times 1 + \frac{1}{2^3} \times 1 + \frac{1}{2^4} \times 1 + \left(1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4}\right) \times 0.5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{5}{32} = \frac{31}{32}$$

$Daughter\,Proportion = 50\%$

3. Baseball Championship

a)
P_i = Probability that the inferior team wins; X = Number of games played:

$$P_i[X = x] = \binom{x-1}{3}(1-p)^3 \times p^{x-1-3} \times (1-p)$$

NB: $X \in [[4 ; 7]]$

Thus: $P_i = \sum_{x=4}^{7} P_i[X = x] = \sum_{x=4}^{7} \binom{x-1}{3}(1-p)^4 \times p^{x-4}$

$P_i = (1-p)^4 \times (1 + 4p + 10p^2 + 20p^3)$

b)
P'_i = Probability that the inferior team wins; X = Number of games played:

Similarly, $P_i'[X = x] = \binom{x-1}{2}(1-p)^2 \times p^{x-1-2} \times (1-p)$, with $X \in [[3 ; 5]]$

Then: $P_i' = \sum_{x=3}^{5} P_i[X = x] = \sum_{x=3}^{5} \binom{x-1}{2}(1-p)^3 \times p^{x-3}$

$P_i' = (1-p)^3 \times (1 + 3p + 6p^2)$

c)
P''_i = Probability that the inferior team wins; X = Number of games played:

In the same fashion, $P_i''[X = x] = \binom{x-1}{4}(1-p)^4 \times p^{x-1-4} \times (1-p)$, with $X \in [[5 ; 9]]$

Then: $P_i'' = \sum_{x=5}^{9} P_i[X = x] = \sum_{x=5}^{9} \binom{x-1}{4}(1-p)^5 \times p^{x-5}$

$P_i'' = (1-p)^5 \times (1 + 5p + 15p^2 + 35p^3 + 70p^4)$

The trend is obvious. We'll also notice that the higher the number of matches, the smaller is the probability improvement from a series with n matches to a series with (n+1) matches.

d)
P_7 = Probability that seven matches are required to determine the winner
P_7 = Probability of one team winning exactly three matches out of six:

$P_7 = \binom{6}{3}0.5^3 \times 0.5^3 = 0.3125$

Assumption: normal distribution, $\sigma = 0.05$
Random variable X: Number of series going to seven matches:

$E[X] = 94 \times 0.3125 = 29.375 \qquad \sigma^2 = 94 \times 0.3125 \times (1 - 0.3125) = 20.195$

Hypothesis testing:

$z = \dfrac{x - \mu}{\sigma} = \dfrac{29.375 - 35}{\sqrt{20.195}} = -1.2517$

We cannot reject the null hypothesis with a 95% confidence level.

4. Birthdays

Method: Determine the probability P_0 that nobody has the same birthday date. Let n be the number of people in the class.

This is obviously a Bernoulli trial. We suppose that nobody is born a February 29.

The right solution is in fact the following:
$$P_0 = \prod_{i=0}^{n-1}\left(1-\frac{i}{365}\right) = \frac{365!}{365^n (365-n)!}$$
And the probability we are looking for is still $1-P_0$.

NB: If I mistakenly stated that *this* solution (the right one) was an approximation, come and see me at the end of the class, and I'll give you full credits for this question. Sorry for the confusion.

5. Guru

Easy enough: let us suppose that the probabilities of the outcomes of each event are approximately equal. The "guru" has only to send his first "prediction" to a large group of people. One half of this group will receive e-mail saying that outcome A_1 will occur, the other half will receive e-mail saying that outcome B_1 is going to happen.

Then, the "guru" divide the group which received the right "prediction" in two half and send the second "prediction", proceeding as before.

In the end, 1 out of 512 people will have received 10 consecutive right "predictions".