Trellis Codes

Lecture 12

Vladimir Stojanović

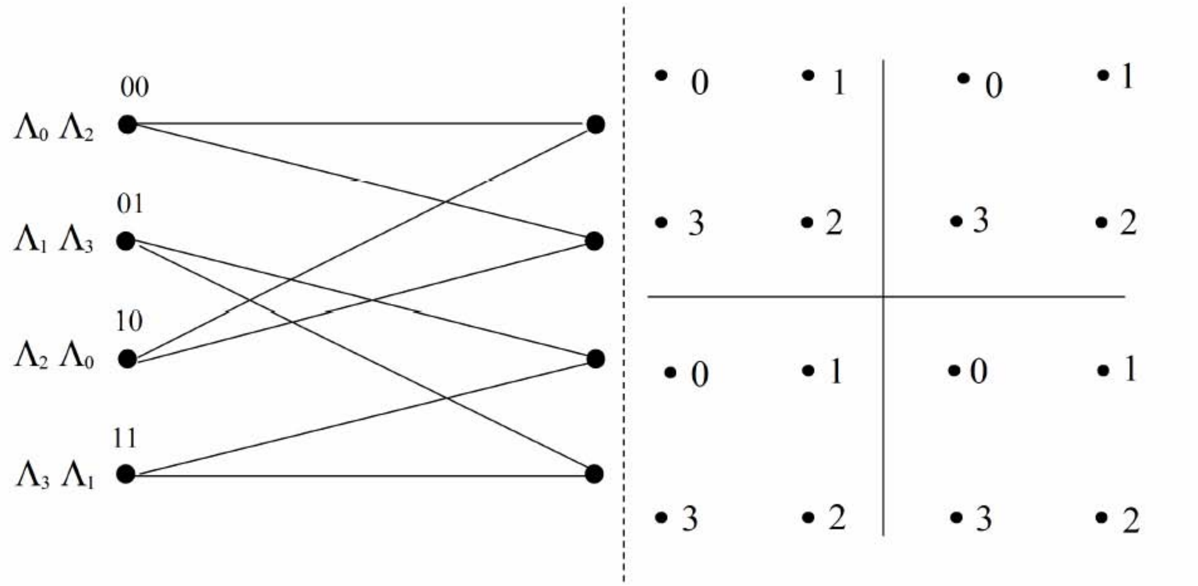
6.973 Communication System Design – Spring 2006

Massachusetts Institute of Technology

Trellis codes

- Invented by Gottfried Ungerboeck of IBM in 1982
- [1] G. Ungerboeck "Channel coding with multilevel/phase signals," *IEEE Transactions on Information Theory,* vol. 28, no. 1, pp. 55-67, 1982.
- [2] G. Ungerboeck "Trellis-coded modulation with redundant signal sets Part II: State of the art," *IEEE Communications Magazine,* vol. 25, no. 2, pp. 12-21, 1987.
- [3] G. Ungerboeck "Trellis-coded modulation with redundant signal sets Part I: Introduction," *IEEE Communications Magazine,* vol. 25, no. 2, pp. 5-11, 1987.

4-state Ungeroboeck Trellis code

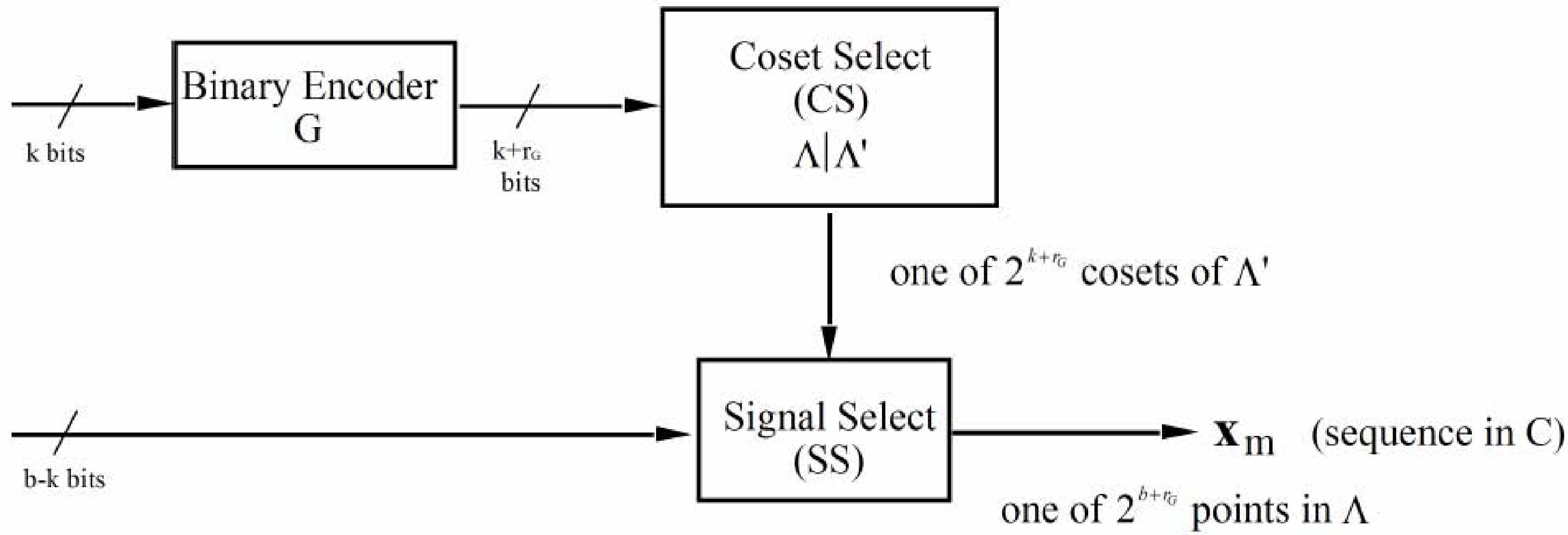


- 1 bit controls the subset (input to conv. encoder)
  - 2 bits choose a point in a subset
- Two minimum distance scenarios
  - Distance between two points in a subset (2 times greater than uncoded 8SQ QAM)
  - When two sequences differ in more than one symbol period
    - Symbol points either chosen from even or odd subsets
    - Within the odds or evens distance the same as 8SQ QAM
      - Diverging at one state and merging at another sate forces the squared distance to be doubled $d_{8SQ}^2 + d_{8SQ}^2 = 2\ d_{8SQ}^2$
- So, this code 3dB better than uncoded 8SQ QAM transmission

Trellis codes – Motivation

- In multi-level modulations
  - Trellis codes allow code design directly for maximization of Euclidean distance
- Hamming distance maximizes Euclidean distance only in binary modulation

General coset (subset) encoder



- x_m – N dimensional vector sequence of points
  - Each N-dimensional symbol chosen from N-dim constellation
  - Sequences of x_m are the codewords $x(D)=sum_m(x_m D^m)$
- Signal constellation has 2^{b+rg} signal points in some coset of N-dimensional real lattice Λ
- Signal constellation contains 2^{k+rg} cosets, each with 2^{b-k} points
- rg_bar=rg/N – normalized redundancy
- kg_bar=k/N – informativity of the coset code

Coset partitioning

- Coset partitioning $\Lambda | \Lambda'$
  - Partition of the lattice Λ into $|\Lambda| \Lambda'|$ (called the "order" of the partition) cosets of a sublattice Λ' such that each point in the original lattice Λ is contained in one, and only one, coset of the sublattice Λ'
- If the encoder G is
  - Convolutional encoder
    - The set of all possible transmitted sequences {x(D)} is a **Trellis Code**
  - Block encoder
    - The set of N-dimensional vectors is a **Lattice Code**
- Both trellis codes and lattice codes are coset codes

Gain of coset codes

- The fundamental gain always with respect to the uncoded system (x_tilda)

$$\gamma_f = \frac{\frac{d^2_{min}(\boldsymbol{x})}{\mathcal{V}_{\boldsymbol{x}}^{2/N}}}{\frac{d^2_{min}(\tilde{\boldsymbol{x}})}{\mathcal{V}_{\tilde{\boldsymbol{x}}}^{2/N}}} = \frac{\frac{d^2_{min}(\boldsymbol{x})}{2^{2(\bar{b}+\bar{r}_G)}\mathcal{V}^{2/N}(\Lambda)}}{\frac{d^2_{min}(\tilde{\boldsymbol{x}})}{2^{2\bar{b}}\cdot\mathcal{V}^{2/N}(\Lambda)}} = \frac{\frac{d^2_{min}(C)}{2^{2\bar{r}_G}\cdot\mathcal{V}^{2/N}(\Lambda)}}{\frac{1}{1}} = \frac{d^2_{min}(C)}{\mathcal{V}(\Lambda)^{2/N}2^{2\bar{r}_G}}$$

- Latice redundancy $\mathcal{V}(\Lambda) = 2^{r_\Lambda} = 2^{N\bar{r}_\Lambda}$

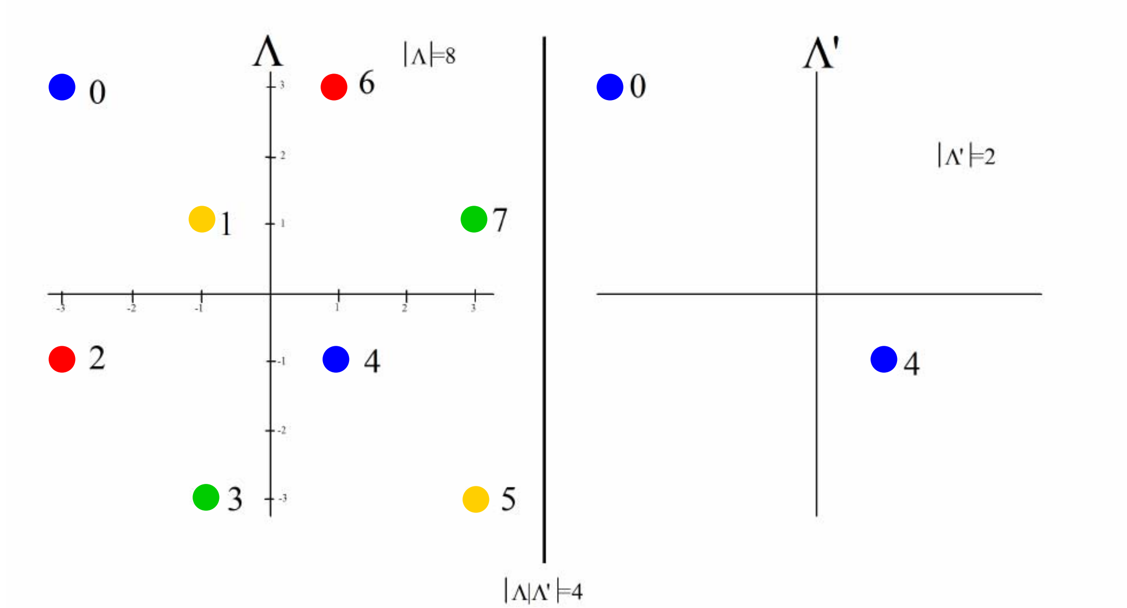
$$\gamma_f = \frac{d^2_{min}(C)}{2^{2(\bar{r}_G+\bar{r}_\Lambda)}} = \frac{d^2_{min}(C)}{2^{2\bar{r}_C}}$$

$$\bar{r}_C = \bar{r}_G + \bar{r}_\Lambda \quad .$$

- Coding gain between 3 and 6dB
- Shaping gain ~1.5dB (fixed by constellation geometry)

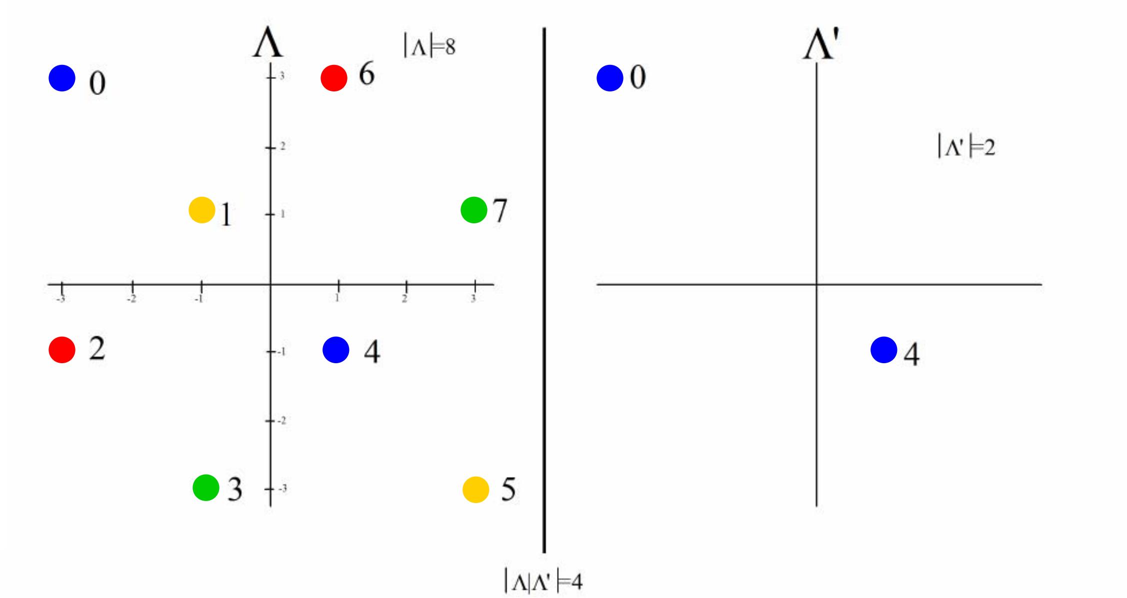
$$\gamma_s = \frac{\mathcal{V}^{2/N}(\Lambda)\cdot 2^{2\bar{r}_G}}{\bar{\mathcal{E}}(\Lambda)} / \frac{1}{(2^{2\bar{b}}-1)/12} = \frac{2^{2\bar{r}_C}}{12\bar{\mathcal{E}}}(2^{2\bar{b}}-1)$$

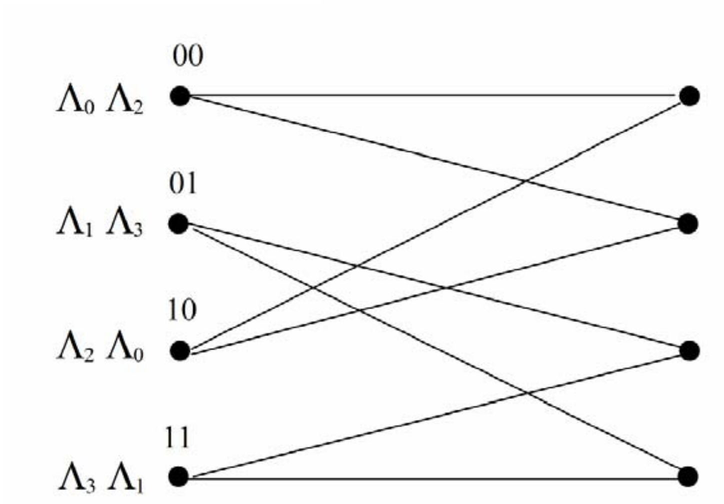
Coset partitioning example (D_2 lattice)

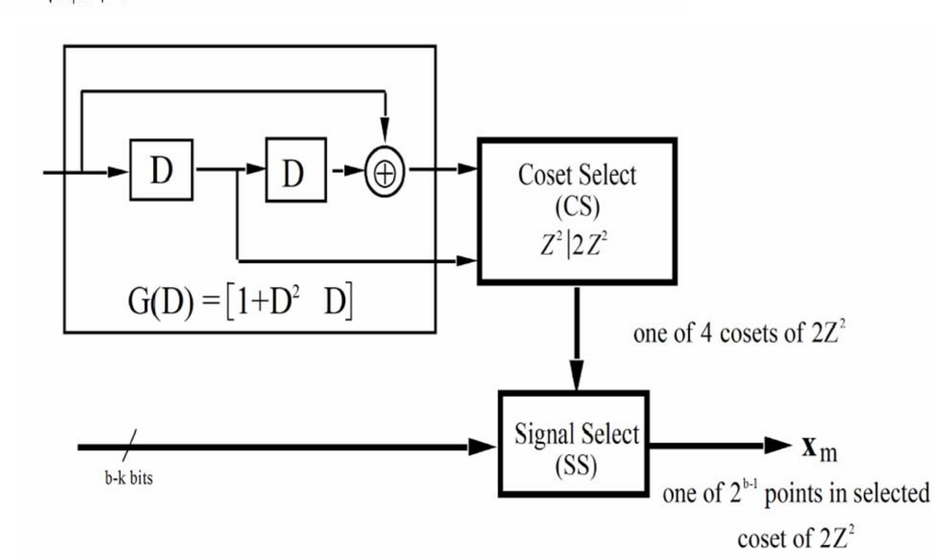


- Ungerboeck rate ½ 3dB trellis code
  - 8AMPM (or 8CR) constellation is a subset of $\Lambda=D_2$ lattice that contains $|\Lambda|=8$ points
  - Average energy per symbol is E=10 (E_bar=5)
  - Sublattice Λ' has a coset Λ_0 with two points $|\Lambda_0|=2$ so that $|\Lambda|\ \Lambda'|=4$ cosets of Λ' in Λ
  - $\Lambda_0=\{0.4\}$ $\Lambda_1=\{1,5\}$ $\Lambda_2=\{2,6\}$ $\Lambda_3=\{3,7\}$
  - These cosets selected by two bit, rate ½ convolutional encoder output

$$G(D) = \begin{bmatrix} 1+D^2 & D \end{bmatrix}$$


Λ
|Λ|=8
Λ'
|Λ'|=2
|Λ|Λ'|=4
0
1
2
3
4
5
6
7


00
01
10
11
Λ0 Λ2
Λ1 Λ3
Λ2 Λ0
Λ3 Λ1


D
D
G(D) = [1+D² D]
Coset Select
(CS)
Z²|2Z²
one of 4 cosets of 2Z²
b-k bits
Signal Select
(SS)
x m
one of 2^(b-1) points in selected
coset of 2Z²

Example, continued

- Min. distance in cosets $d_{min}(\Lambda')=2d_{min}(\Lambda)=4\text{sqrt}(2)$
- Sequence distance (any two paths that start and terminate in the same pair of states must have a distance that is d'=sqrt(16+8+16) >= 4sqrt(2)
- So, the parallel transition distance is the minimum distance for this code
  - This is still sqrt(2) better than distance corresponding to no extra bit (or just transmitting uncoded 4QAM)

$$\gamma = \frac{(d^2_{min}/\mathcal{E}_{x_p})_{\text{coded}}}{(d^2_{min}/\mathcal{E}_{x_p})_{\text{uncoded}}} \qquad \gamma = \frac{\frac{16\cdot 2}{10}}{\frac{1}{1/2}} = 1.6 = 2 \text{ dB}$$

The fundamental coding gain is (realizing that $\bar{r}_C = \bar{r}_\Lambda + \bar{r}_G = 1.5 + .5 = 2$)

$$\gamma_f = \left(\frac{d^2_{min}}{2^{2\bar{r}_C}}\right) = \frac{32}{2^{2\cdot 2}} = 2 \text{ (3 dB)} \quad .$$

$$\gamma_s = \frac{2^{2\cdot 2}}{12\cdot 5}\left(2^2 - 1\right) = \frac{4}{5} = \text{-1 dB}$$

Mapping by set partitioning

- Basic partitioning can be extended systematically to larger values of b (i.e. constellation sizes)

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- Ungerboeck labeling in two dimensions
  - The LSB v_0 of the encoder output is used to specify which of the first 2 partitions (B_0, v_0=0 or B_1, v_0=1) contains the selected coset of the sublattice Λ', and then uses v1 to specify which of the next level parititions (C_0,C_2,C_1,C_3) contains the selected coset of the sublattice, etc.
  - The remaining bits v_{k+r}, …, v_{b+r-1} are used to select points within the coset
- In practice, this mapping is often used for N=1,2,4 and 8
  - One dimensional partitioning halves PAM constellation into sets of “every other point”, realizing 6dB increase in intra-partition distance for each such halving
    - In 4 and 8 dimensions the distance is 1.5dB and 0.75dB per partition, respectively)

8PSK mapping by set partitioning

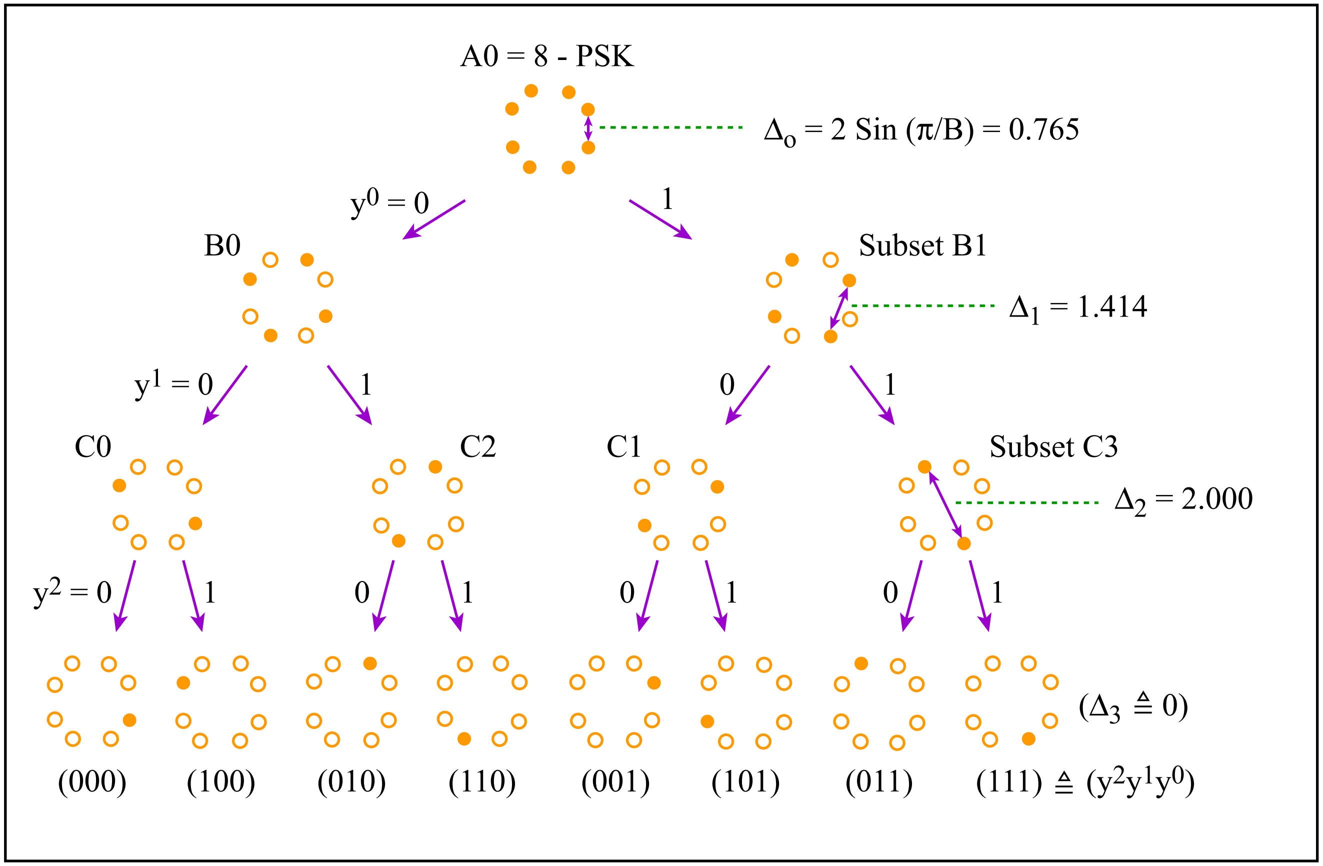

Figure by MIT OpenCourseWare.

- Ungerboeck

PSK example from Ungerboeck

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- Path distance greater than internal coset distance $\Delta_2=2$, so
  - d_{free}=min(path distance, internal coset distance)=2

Benefiting from larger number of states

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BER improvement

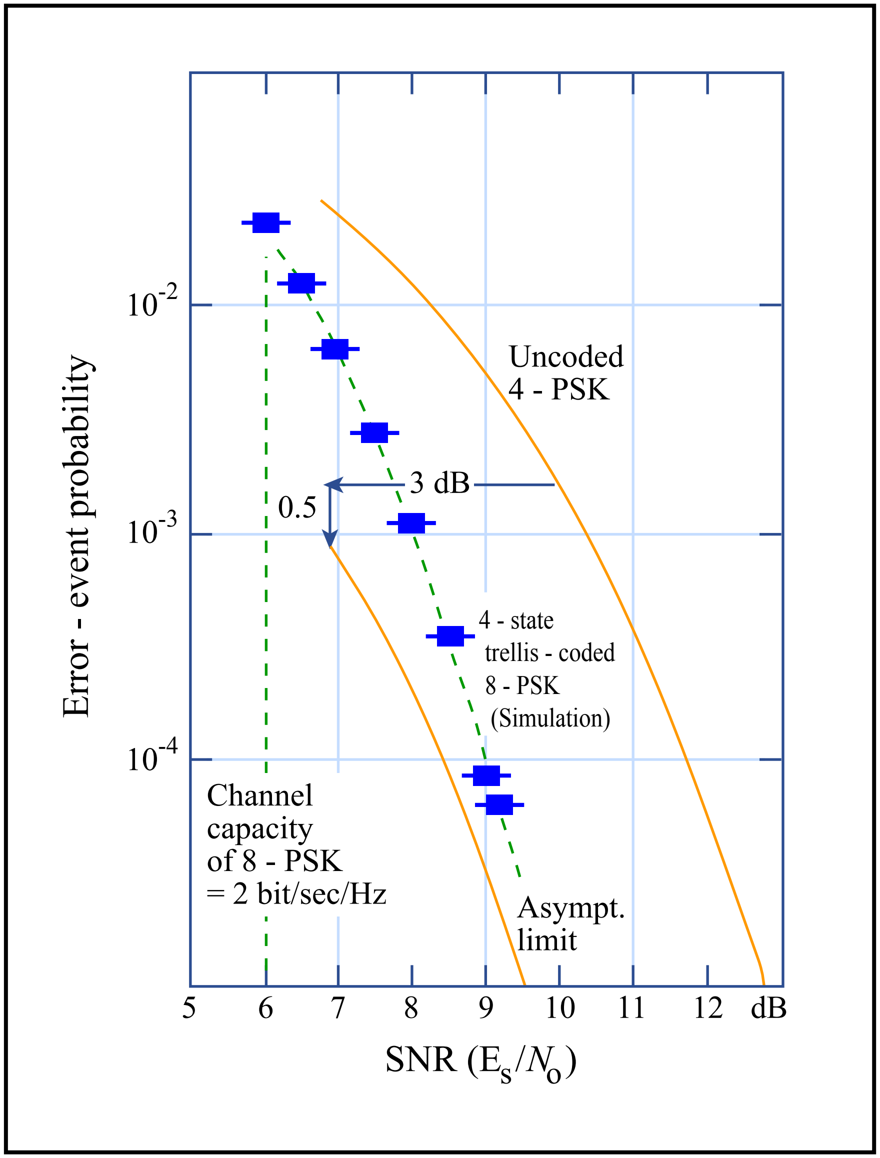

Figure by MIT OpenCourseWare.

QAM example

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- For m information bits need 2^{m+1} points
  - Extra bit chooses even or odd cosets
  - Coding gain of approx 4dB over uncoded modulation

Trellis for QAM example

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- Error paths with distance $5d_0^2$ from sequence D0-D0-D3-D6
  - All error paths start and re-emerge in one node

Coding gain vs. state

- Significant gains
  - With as few as 4,8,16 states
  - 3dB (4 states)
  - 4dB (8 states)
  - 5dB (16 states)
  - up to 6dB (128 or more)
- Doubling of states does not always increase dfree
  - Can get big increase in
  - Num. nearest neighbors
  - Num. next-nearest neighbors

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Another example

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- Rate 2/3 trellis code
  - Increase fundamental gain beyond 3dB (which was the parallel transition distance in the rate ½ code)
  - Need constellation partitioning by one additional level/step to ensure that the parallel transition distance will now be 6dB

Min. distance

- Now, min distance occurs between two longer length sequences through the trellis, instead of between parallel transitions

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One-dimensional TCM

- Up to 6dB of fundamental gain $\gamma_f \leq 16/(2^{2 \cdot 1}) = 6 \text{ dB}$
  - Only need to partition twice $\Lambda' = \Lambda_{(2)}$ to realize min separation between any two parallel transitions that is 6dB higher than uncoded PAM
  - The partition chain for the one-dimensional trellis codes is, with $\Lambda = Z$, $Z|2Z|4Z$, with corresponding min distances between points $d_{min}(Z)=1$, $d_{min}(2Z)=2$, $d_{min}(4Z)=4$, and $r_G=1$
  - The parallel separation is never more than $d^2=16$
  - G(D) must then be a rate ½ code

One-dimensional Trellis code tables

2^{ν}	h_1	h_0	d^2_{min}	γf	(dB)	$\bar{N}_e$	$\bar{N}_1$	$\bar{N}_2$	$\bar{N}_3$	$\bar{N}_4$	$\tilde{\gamma} f$	$\bar{N}_D$
4	2	5	9	2.25	3.52	4	8	16	32	64	3.32	12
8	04	13	10	2.50	3.98	4	8	16	40	72	3.78	24
16	04	23	11	2.75	4.39	8	8	16	48	80	3.99	48
16	10	23	11	2.75	4.39	4	8	24	48	80	4.19	48
32	10	45	13	3.25	5.12	12	28	56	126	236	4.60	96
64	024	103	14	3.50	5.44	36	0	90	0	420	4.61	192
64	054	161	14	3.50	5.44	8	32	66	84	236	4.94	192
128	126	235	16	4.00	6.02	66	0	256	0	1060	5.01	384
128	160	267	15	3.75	5.74	8	34	100	164	344	5.16	384
128	124	207	14	3.50	5.44	4	8	14	56	136	5.24	384
256	362	515	16	4.00	6.02	2	32	80	132	268	5.47	768
256	370	515	15	3.75	5.74	4	6	40	68	140	5.42	768
512	0342	1017	16	4.00	6.02	2	0	56	0	332	5.51	1536

	A_1^0			
d_{min} w.r.t. A_1^0	1			
N_e w.r.t. A_1^0	2			
	B_1^0		B_1^1	
d_{min} w.r.t. B_1^0	2		1	
N_e w.r.t. B_1^0	2		2	
	C_1^0	C_1^2	C_1^1	C_1^3
d_{min} w.r.t. C_1^0	4	2	1	1
N_e w.r.t. C_1^0	2	2	1	1

Figure by MIT OpenCourseWare.

- Ne – number of nearest neighbors
- $N_{1,2,3,4}$ numbers of next-to-near neighbors

Two dimensional codes

- Use 3-level partitioning so that $\Lambda' = \Lambda_{(3)}$
  - To realize min separation between any two parallel transitions that is 6dB higher than uncoded two-dimensional QAM
  - The partition chain is with $\Lambda = Z^2$, $Z^2|D_2|2Z^2|2D_2$
  - Corresponding min distance $d_{min}(Z^2)=1$, $d_{min}(D_2)=sqrt(2)$, $d_{min}(2Z^2)=2$, $d_{min}(2D_2)=2sqrt(2)$ and $r_G=1$
  - $r_G=1$ implies doubling of the two-dimensional constellation size $|\Lambda|$ with respect to uncoded transmission. the maximum fundamental gain is limited to $\gamma_f \leq 8/2 = 6\text{dB}$

Two-dimensional partitioning

2^ν	h_2	h_1	h_0	d^2_{min}	γf	(dB)	$\bar{N}_e$	$\bar{N}_1$	$\bar{N}_2$	$\bar{N}_3$	$\bar{N}_4$	$\tilde{\gamma} f$	$\bar{N}_D$
4	-	2	5	4	2	3.01	2	16	64	256	1024	3.01	8
8	04	02	11	5	2.5	3.98	8	36	160	714	3144	3.58	32
16	16	04	23	6	3	4.77	28	80	410	1952	8616	4.01	60
32	10	06	41	6	3	4.77	8	52	202	984	4712	4.37	116
32	34	16	45	6	3	4.77	4	64	202	800	4848	4.44	116
64	064	016	101	7	3.5	5.44	28	130	504	2484	12236	4.68	228
64	060	004	143	7	3.5	5.44	24	146	592	2480	12264	4.72	228
64	036	052	115	7	3.5	5.44	20	126	496	2204	10756	4.78	228
128	042	014	203	8	4	6.02	172	0	2950	0	73492	4.74	451
128	056	150	223	8	4	6.02	86	312	1284	6028	29320	4.94	451
128	024	100	245	7	3.5	5.44	4	94	484	1684	8200	4.91	451
128	164	142	263	7	3.5	5.44	4	66	376	1292	6624	5.01	451
256	304	056	401	8	4	6.02	22	152	658	2816	13926	5.23	900
256	370	272	417	8	4	6.02	18	154	612	2736	13182	5.24	900
256	274	162	401	7	3.5	5.44	2	32	124	522	2732	5.22	900
512	0510	0346	1001	8	4	6.02	2	64	350	1530	6768	5.33	1796

	A_2^0			
d_{min} w.r.t. A_2^0	1			
N_e w.r.t. A_2^0	4			
	B_2^0		B_2^1	
d_{min} w.r.t. B_2^0	$\sqrt{2}$		1	
N_e w.r.t. B_2^0	4		4	
	C_2^0	C_2^2	C_2^1	C_2^3
d_{min} w.r.t. C_2^0	2	$\sqrt{2}$	1	1
N_e w.r.t. C_2^0	4	4	2	2
	D_2^0	D_2^2	D_2^1	D_2^3
d_{min} w.r.t. D_2^0	$\sqrt{8}$	$\sqrt{2}$	1	1
N_e w.r.t. D_2^0	4	2	1	1
	D_2^4	D_2^6	D_2^5	D_2^7
d_{min} w.r.t. D_2^0	2	$\sqrt{2}$	1	1
N_e w.r.t. D_2^0	4	2	1	1

Figure by MIT OpenCourseWare.

- Notice larger number of Ne and $N_{1,2,3,4}$