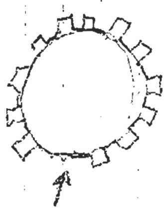
6.866 10/26 D412

Iterative Modification

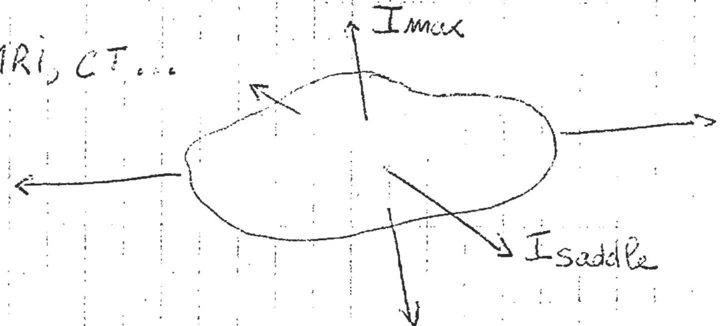
An application of swelling / thining: detect missing teeths on a pair

 → swelling → thining → 

3 swell, 6 thin, 3 swell...

Another application: detect contours error, etc...

3D: MRI, CT...

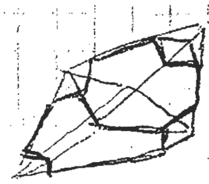


3D Connectivity problem ... cubical voxels!

6 face adj.
12 edge adj.
8 vertex adj.
26

} none of them is a good connectivity

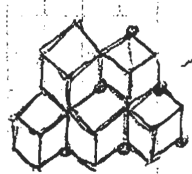
Tesselation of space: semi-regular → truncated octohedron



8 hexagons
6 squares
14 faces

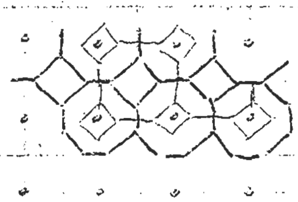
rhombic dodecahedron : all 12 faces have the same area.

- How does it tesselate space? Go back to 2D :



→ add layers → 3D (w/ offset)

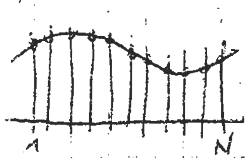
you can also keep square tesselation and change def. of connectivity



octahedron seen from the top.

<u>Calculus of Variation (COV)</u> [A6]

finite # parameters → calculus <u>BUT</u> what if we have an unkown function (i.e. unknown # parameters)



What if $N \to \infty$?

<u>Example I:</u> shape from gradient

$$\min_{z(x,y)} \iint (p(x,y) - z_x(x,y))^2 + (q(x,y) - z_y(x,y))^2 \, dx\,dy$$

p and q : known   z : unknown   In practice, there is an error.

Finding z is a COV-problem. Usually, we discretize. messy...

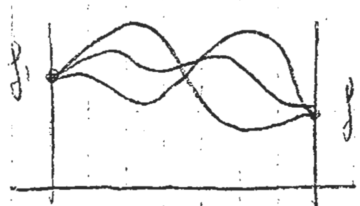
<u>Example II:</u> optical flow

Example : simple

$$I = \int_{x_1}^{x_2} F(x; y, y')\, dx$$

$y(x_1) = y_1$

$y(x_2) = y_2$



- Assume $y(x)$ is a solution
- Make a "small variation"
- Add a test function $\eta(x)$

Any variation makes things worse...

$y(x) + \varepsilon\, \eta(x)$. We expect a quadratic dependence on ε .

$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$ for <u>all</u> $\eta(x)$ such as $\eta(x_1) = \eta(x_2) = 0$.

$y'(x) \to y'(x) + \varepsilon\, \eta'(x)$ Thus $I \to \int_{x_1}^{x_2} F(x; y + \varepsilon \eta; y' + \varepsilon \eta')\, dx$

See appendix A6 for details :

$\int_{x_1}^{x_2} (\eta(x) F_y + \eta' F_{y'})\, dx = 0$ for all $\eta(x)$

Suppose we take : $\eta(x) = \delta(x - x_0) \to \int_{x_1}^{x_2} \eta(F_y + F_{y'})\, dx = 0$ WRONG!

But integration by parts is RIGHT! ($\int u\, dv = uv\Big]_a^b - \int v\, du$)

$$\int_{x_1}^{x_2} \eta' F_{y'}\, dx = \underbrace{\eta(x) F_{y'} \Big]_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} F_{y'}\, dx$$

Thus : $\int_{x_1}^{x_2} \eta(x) \left[F_y - \frac{d}{dx} F_{y'} \right] dx = 0$ for <u>all</u> $\eta(x)$

Thus : $\boxed{F_y - \frac{d}{dx} F_{y'} = 0}$ EULER EQUATION example:

$$\min_{f(x)} \int_{x_1}^{x_2} \sqrt{1+f'(x)^2}\, dx$$

$$F_f = 0 \qquad F_{f'} = \frac{f'(x)}{\sqrt{1+f'(x)^2}} \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+f'(x)^2}}\right) = 0$$

$$\frac{f'(x)}{\sqrt{1+f'(x)^2}} = k \quad \Rightarrow \quad f'(x) \text{ is a constant !} \qquad \underline{f(x) = mx + c}$$

Simple because:

1. boundary conditions
2. first derivative only
3. one function
4. single independent variable

General:

1. $[\, y(x)\, F_{f'} \,]_{x_1}^{x_2} = 0$ for all $y(x)$

   $F_{f'} = 0$ at x_1 and x_2

   "natural boundary conditions"

2. higher derivatives

$$I = \int_{x_1}^{x_2} F(x, f, f', f'', \dots)\, dx \quad \Rightarrow \quad \boxed{F_f - \frac{d}{dx} F_{f'} + \frac{d}{dx^2} F_{f''} - \dots = 0}$$

B.C on all but highest derivatives

3. More than one function $u(x,y)$ $v(x,y)$

$$I = \int_{x_1}^{x_2} F(x, f_1, f_2, f_1', f_2', \dots)\, dx = 0 \qquad \boxed{F_{f_i} - \frac{d}{dx} F_{f_i'} = 0}$$

4. Two indep. variables x y

$$I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x, y, f, f_x, f_y)\, dx\, dy \qquad \text{boundary } \partial D$$

Same plan $\eta(x,y)$ $\quad \iint_D (\eta F_f + \eta_x F_{f_x} + \eta_y F_{f_y})\, dx\, dy = 0$

Note:

Gauss's integral theorem $\quad \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\, dy = \oint_{\partial D} Q\, dy - P\, dx$

Thus:

$$\iint \sim\sim = \oint_{\partial D} (\eta F_{f_x}\, dy - \eta F_{f_y})\, dx = 0 \qquad (\eta = 0 \text{ on } \partial D)$$

By expanding $\iint_D \frac{\partial}{\partial x}(\eta F_{f_x}) + \frac{\partial}{\partial y}(\eta F_{f_y})\, dx\, dy$, we get:

$\iint_D \eta \left(F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y}\right) dx\, dy = 0$ for all η thus

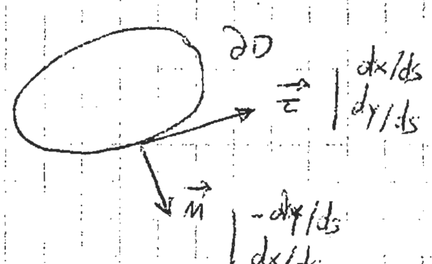
$$\boxed{F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0} \quad \text{2D EULER EQUATION (PDE)}$$

Note: what if we don't know B.C.?

we need $\int_{\partial D} \eta (\ldots) = 0$

W/O B.C, "natural BC" is: $F_{f_x} \frac{dy}{ds} = F_{f_y} \frac{dx}{ds}$ on ∂D

or, $\quad (F_{f_x}, F_{f_y}) \cdot \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{ds}\right) = 0$



Example: Shape from gradient $\quad p(x,y) \quad q(x,y)$

$$\min_{z(x,y)} \iint_D \underbrace{(p(x,y)-z_x(x,y))^2 + (q(x,y)-z_y(x,y))^2}_{F} \, dx\, dy$$

Euler $\Rightarrow \quad F_z - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0 \qquad$ where f is z

$$0 - \frac{\partial}{\partial x}\left(-2(p-z_x)\right) - \frac{\partial}{\partial y}\left(-2(q-z_y)\right) = 0$$

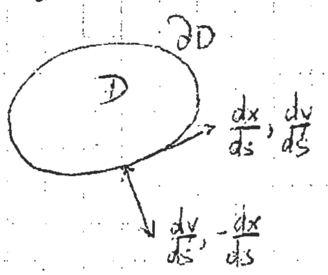
Thus: $\quad \underline{\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = z_{xx} + z_{yy}} \quad \nabla^2 z$ Laplacian

$z_x = p$? $\; z_y = q$? Hard to satisfy because of noise, too many constraints

This is an over-determined problem.

B.C.? $\qquad$ "Natural" $\quad (F_{z_x}, F_{z_y}) \cdot \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) = 0$

normal



$(p - z_x, q - z_q) \cdot (\ ,\ ) = 0$

$(p, q) \cdot \hat{n} = (z_x, z_y)\hat{n} \qquad$ (sanity check)

Other example: