Lumped-element Modeling with Equivalent Circuits

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Outline

> **Context and motivation**

> **Lumped-element modeling**

> **Equivalent circuits and circuit elements**

> **Connection laws**

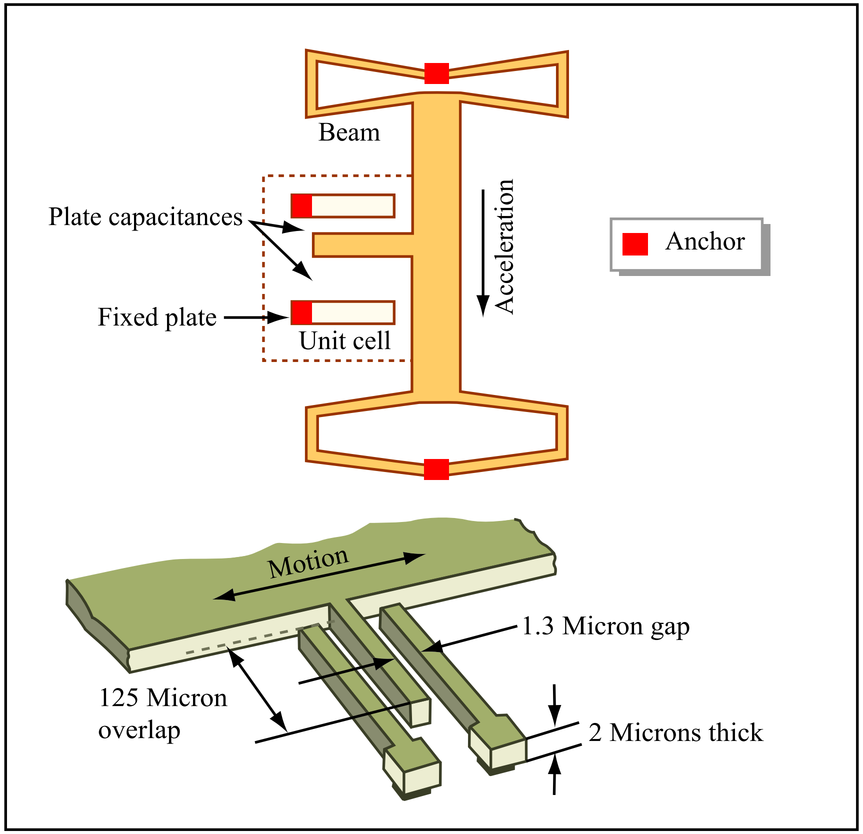
Context

- **Where are we?**
  - **We have just learned how to make structures**
  - **About the properties of the constituent materials**
  - **And about elements in two domains**
    - **structures and electronics**
- **Now we are going to learn about modeling**
  - **Modeling for arbitrary energy domains**
  - **How to exchange energy between domains**
    - **Especially electrical and mechanical**
  - **How to model dynamics**
- **After, we start to learn about the rest of the domains**

Inertial MEMS

> Analog Devices Accelerometer

- **ADXL150**
- **Acceleration ➔ Changes gap ➔ capacitance ➔ electrical output**



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Photograph of a circuit board.

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Micrograph of machined microchannels.

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RF MEMS

> Use electrical signal to create mechanical motion

> Series RF Switch (Northeastern & ADI)

- Cantilever closes circuit when actuated → relay

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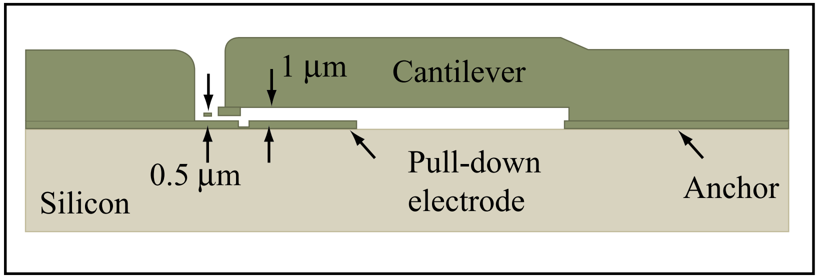
Figure 11 on p. 342 in: Zavracky, P. M., N. E. McGruer, R. H. Morrison, and D. Potter. "Microswitches and Microrelays with a View Toward Microwave Applications." *International Journal of RF and Microwave Computer-Aided Engineering* 9, no. 4 (1999): 338-347.



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Adapted from Rebeiz, Gabriel *MRF MEMS: Theory, Design, and Technology.* Hoboken, NJ: John Wiley, 2003. ISBN: 9780471201694.

Zavracky *et al.*, *Int. J. RF Microwave CAE*, 9:338, 1999, via Rebeiz *RF MEMS*

What we'd like to do

- > These systems are complicated 3D geometries
- > Transform electrical energy ←→ mechanical energy
- > How do we design such structures?
  - Multiphysics FEM
    - » Solve constitutive equations at each node
    - » Tedious but potentially most accurate

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Distorted switch (Coventor)

- > Is there an easier way?
  - That will capture dimensional dependencies?
  - Allow for quick iterative design?
  - Maybe get us within 10-20%?

RF MEMS Switch

> **What we'd really like to know**

- **What voltage will close the switch?**
- **What voltage will open the switch (when closed)?**
- **How fast will this happen?**
- **What are the tradeoffs between these variables?**
  - » **Actuation voltage vs. maximum switching frequency**

> **So let's restrict ourselves to relations between voltage and tip deflection**

- **Hah! – we have "lumped" our system**

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- Equivalent circuits and circuit elements
- Connection laws

Lumped-element modeling

> **What is a lumped element?**

- **A <u>discrete</u> object that can <u>exchange energy</u> with other objects**
- **An object whose internal physics can be combined into terminal relations**
- **Whose size is smaller than wavelength of the appropriate signal**
  - » **Signals do not take time to propagate**

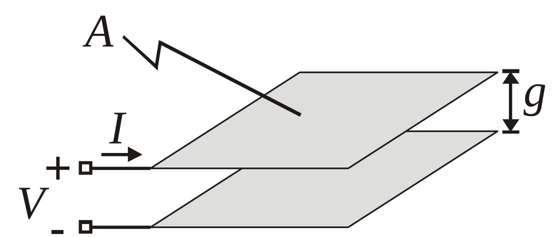
Lumped elements

> **Electrical capacitor**

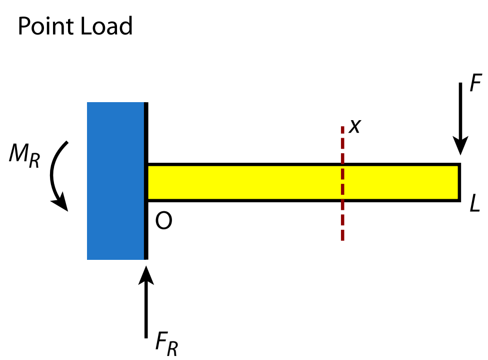
> **Spring**

> **Rigid mass**
- **Push on it and it moves**
- **Relation between force and displacement**

> **Fluidic channel**
- **Apply pressure and fluid flows instantaneously**
- **Relation between pressure and volumetric flow rate**



$$I = C\frac{dV}{dt}$$



$$w_{\max} = \frac{1}{k_{cantilever}} F$$

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Adapted from Figure 9.7 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 209. ISBN: 9780792372462.

Pros/cons of lumped elements

- **Pros**
  - **Simplified representations that carry dimensional dependencies**
  - **Can do equivalent circuits**
  - **Static and dynamic analyses**
- **Cons**
  - **Lose information**
    - **Deflection along length of cantilever**
  - **Will not get things completely right**
    - **Capacitance due to fringing fields**

So how do we go about lumping?

> **First, we need input/output relations**

- **This requires solving physics**
- **This is what we do in the individual domains**
  - » **We have already done this in electrical and mechanical domains**

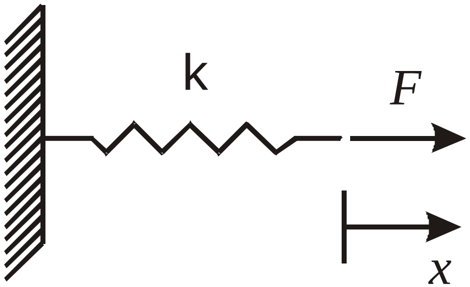
> **For cantilever RF switch**

- **What is relation between force and tip deflection?**
- **Not voltage and deflection**
  - » **Different energy domains**

RF Switch mechanical model

> **We have seen that there is a linear relation between force and tip deflection**

- **Cantilever behaves as linear spring *k***
- **CAVEAT: *k* is specific for this problem**
- **Different *k*'s for same cantilever but**
  - » **Distributed force applied over whole cantilever**
  - » **Point force applied at end**
  - » **Deflection of cantilever middle is needed**
  - » **Etc.**

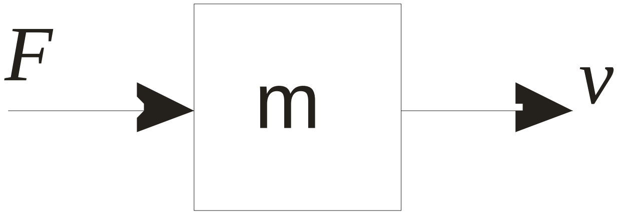


$$F = \mathbf{k}x$$

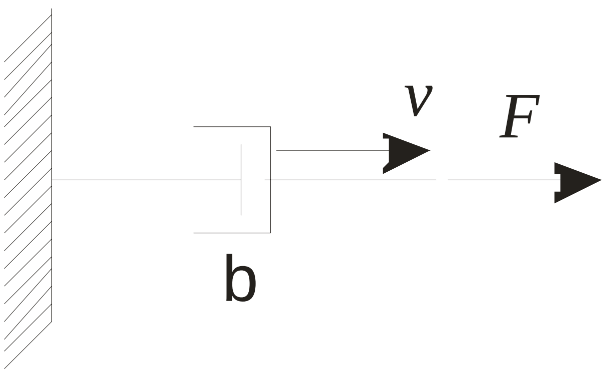
> **Lesson: Don't just use equation out of a book**

RF Switch mechanical model

- What else is needed for model?
- Inertia of cantilever → Lumped mass
- Energy loss → Lumped dashpot
  - Due to air damping



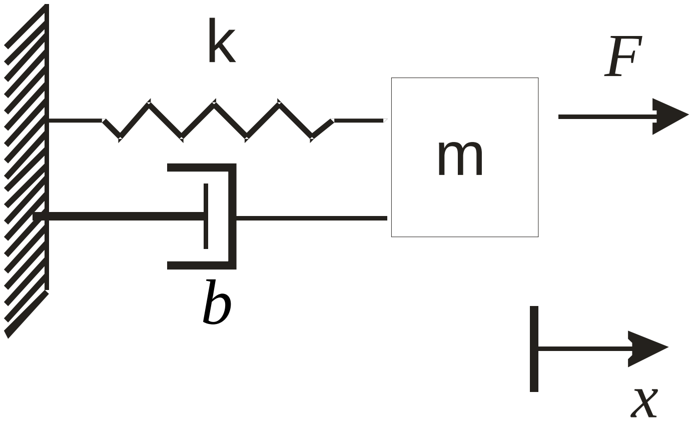
$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$



$$F = bv = b\frac{dx}{dt}$$

How do we connect these together?

- Intuition and physics
- Example: cantilever switch
  - Tip movement (*x*) stretches spring
  - And causes damping
  - Tip has mass associated with it
  - All elements have same displacement



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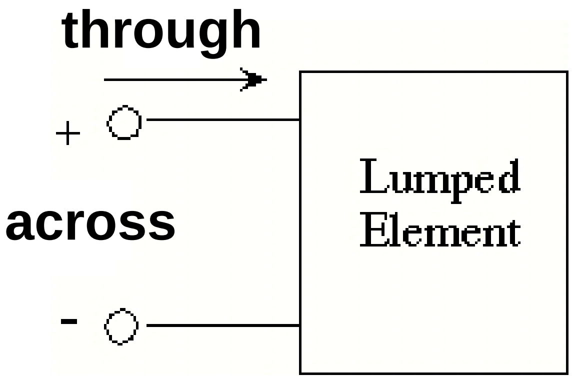
Why use equivalent circuits?

- One modeling approach
  - Use circuits for electrical domain
    - Solve via KCL, KVL
  - Use mechanical lumped elements in mechanical domain
    - Solve via Newton's laws
  - Connect two using ODEs or matrices or other representation
- Our approach
  - Lumped elements have electrical equivalents
  - Can hook them together such that solving circuit intrinsically solves Newton's laws (or continuity relationships)
  - Now we have ONE representation for many different domains
  - VERY POWERFUL

Onward to equivalent circuits

- Each lumped element has one or more ports
- Each port is associated with two variables
  - A "through" variable
  - An "across" variable
- Power <u>into</u> the port is defined by the product of these two variables



Onward to equivalent circuits

- In electrical circuits, voltage is physically “across” and current is physically “through”

  voltage → across

  current → through

- What happens when we *translate* mechanics into equivalent circuits?

force → across (V)	OR	force → through (I)
velocity → through (I)		velocity → across (V)

- Why does this matter?

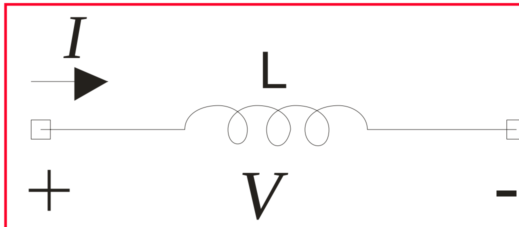
What circuit element is the spring?

- It stores elastic energy
- Is it a capacitor or an inductor?

$$F = kx$$

$$F = k\int \dot{x}\,dt$$

$$L = {}^{1}\!/_{\mathbf{k}}$$



$$V = L\frac{dI}{dt}$$

$$I = \frac{1}{L}\int V\,dt$$

force → through (I)

velocity → across (V)

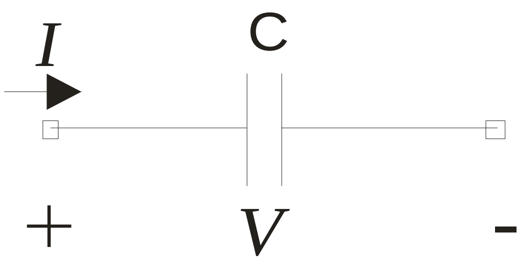
force → across (V)

velocity → through (I)

$$x = {}^{1}\!/_{k}\,F$$

$$\dot{x} = {}^{1}\!/_{k}\,\frac{dF}{dt}$$

$$C = {}^{1}\!/_{\mathbf{k}}$$



$$I = C\frac{dV}{dt}$$

Which is correct?

- Both are correct
- And both are used → beware!
- Velocity → voltage
  - "Indirect" or "mobility" analogy
  - Cleaner match between physical system and circuit
    - Velocity is naturally "across" (e.g., relative) variable
  - But stores mechanical PE in inductors, KE in capacitors
  - Springs → Inductors
- Force → voltage
  - "Direct" analogy
  - Always store PE in capacitors
  - Springs → Capacitors

This is what we will use

- Circuit topologies are dual of each other

Generalized variables

- We want a consistent modeling approach across different domains
- Can we generalize what we just did?
  - YES

Generalized variables

- Formalize "terminal" relations
- Displacement $q(t)$
- Flow $f(t)$: the derivative of displacement
- Effort $e(t)$
- Momentum $p(t)$: the integral of effort
- Net power into device is effort times flow

General	Mechanical
$f = \frac{dq}{dt}$	$v = \frac{dx}{dt}$
$q = q_o + \int_0^t f dt$	$x = x_o + \int_0^t v dt$
$e = \frac{dp}{dt}$	$F = \frac{dp}{dt}$
$p = p_o + \int_0^t e dt$	$p = p_o + \int_0^t F dt$

$$P_{net} = e \cdot f$$

Examples

> **Effort-flow relations occur in MANY different energy domains**

General	Electrical	Mechanical	Fluidic	Thermal
Effort (e)	Voltage, V	Force, F	Pressure, P	Temp. diff., ΔT
Flow (f)	Current, I	Velocity, v	Vol. flow rate, Q	Heat flow
Displacement (q)	Charge, Q	Displacement, x	Volume, V	Heat, Q
Momentum (p)	-	Momentum, p	Pressure Momentum, Γ	-
Resistance	Resistor, R	Damper, b	Fluidic resistance, R	Thermal resistance, R
Capacitance	Capacitor, C	Spring, k	Fluid capacitance, C	Heat capacity, mcp
Inertance	Inductor, L	Mass, m	Inertance, M	-
Node law	KCL	Continuity of space	Mass conservation	Heat energy conservation
Mesh law	KVL	Newton's 2^{nd} law	Pressure is relative	Temperature is relative

Other conventions

- **Thermal convention: T becomes the across variable (voltage) and heat-flow becomes the through variable (current)**
  - **Conserved quantity is heat energy**

Building equivalent circuits

> **Need power sources**

> **Passive elements**

> **Topology and connection rules**

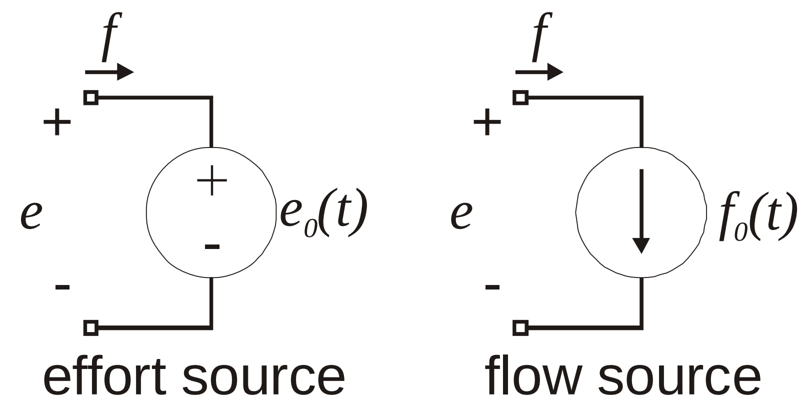
- **Figure out how to put things together**

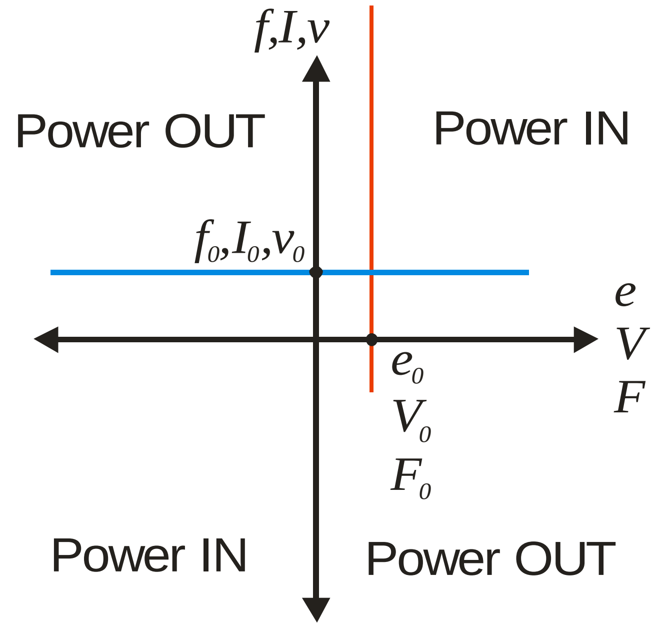
> **What do we get?**

- **An intuitive representation of the relevant physics**
- **Ability to model many domains in one representation**
- **Access to *extremely* mature circuit analysis techniques and software**

One-port source elements

- > **Effort source and flow source**
- > **Effort source establishes a time-dependent effort independent of flow**
  - **Electrical voltage source**
  - **Pressure source**
- > **Flow source establishes a time-dependent flow independent of effort**
  - **Electrical current source**
  - **Syringe pump**



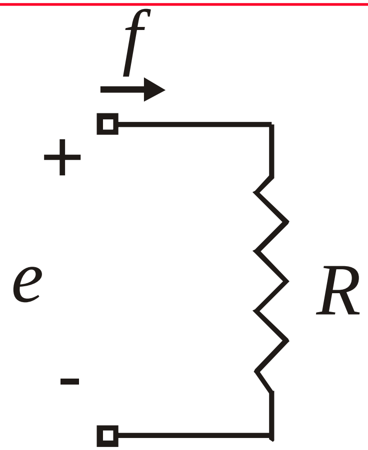


One-port circuit elements

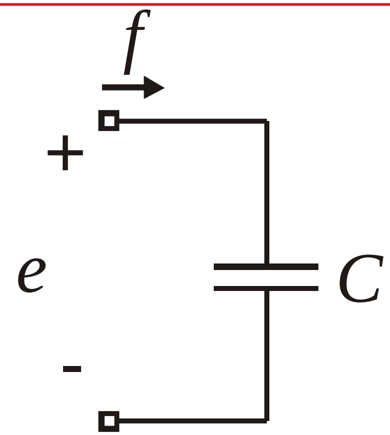
> **Three general passive elements**

> **Represent different functional relationships**

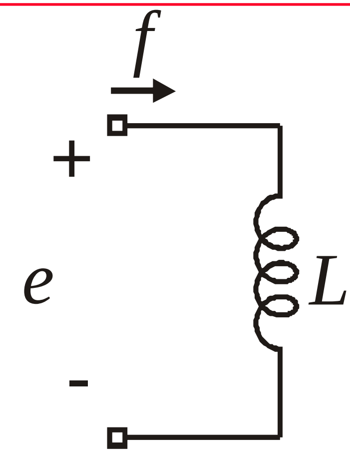
- **Energy storage, dissipation**



Relates e **&** f
Directly relates e **&** f



Relates e **&** q
Differentiates e
Integrates f

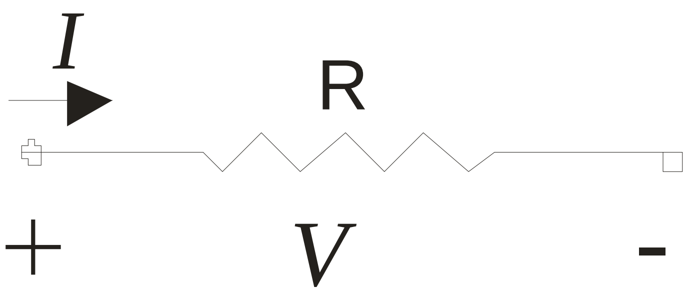


Relates f **&** p
Integrates e
Differentiates f

Analogies between mechanics and electronics

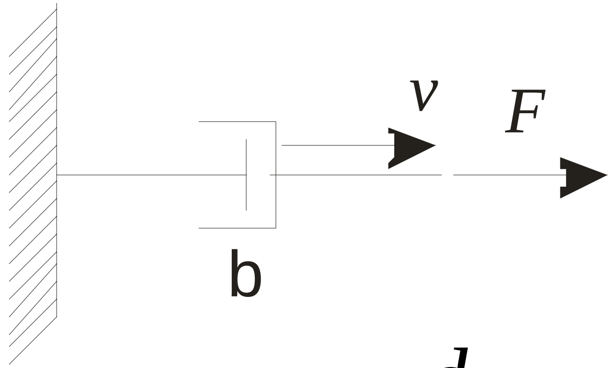
> **Electrical Domain**

- **A resistor**



> **Mechanical Domain**

- **A damper (dashpot)**



$$V = RI = R\frac{dQ}{dt}$$

$$R = b$$

$$F = bv = b\frac{dx}{dt}$$

> **There is again a correspondence between**

- **V and F**
- **I and v**
- **Q and x**
- **R and b**

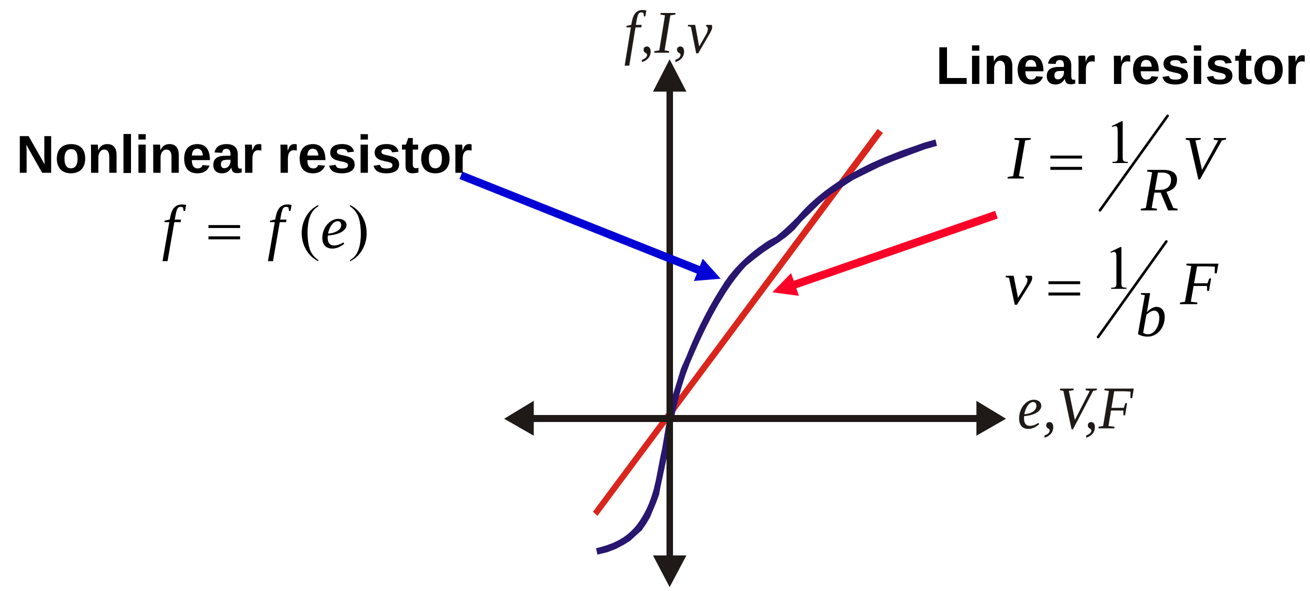
> *Electrical Power* = VI

> *Mechanical Power* = Fv

Generalized resistor

> **For the resistor,**

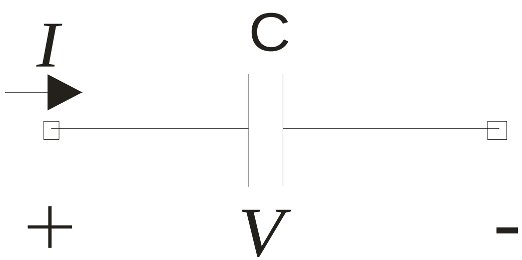
- ***e*** **is an algebraic function of *f* (or vice versa)**
- **Can be a nonlinear function**



Analogies between mechanics and electronics

> **Electrical Domain**

- **A capacitor**

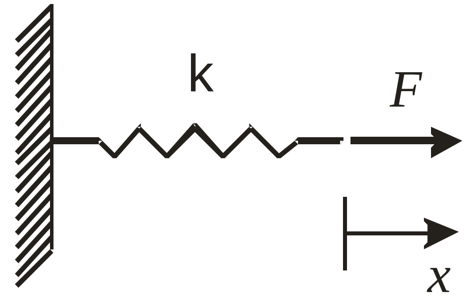


$$Q = CV$$

$$I = C\frac{dV}{dt}$$

> **Mechanical Domain**

- **A spring**



$$x = {}^{1}\!/_{k}\, F$$

$$\frac{dx}{dt} = \dot{x} = {}^{1}\!/_{k}\, \frac{dF}{dt}$$

$$C = \frac{1}{k}$$

> **There is again a correspondence between**

- ***V* and *F***
- ***I* and *v***
- ***Q* and *x***
- ***R* and *b***

> *Electrical Power = VI*

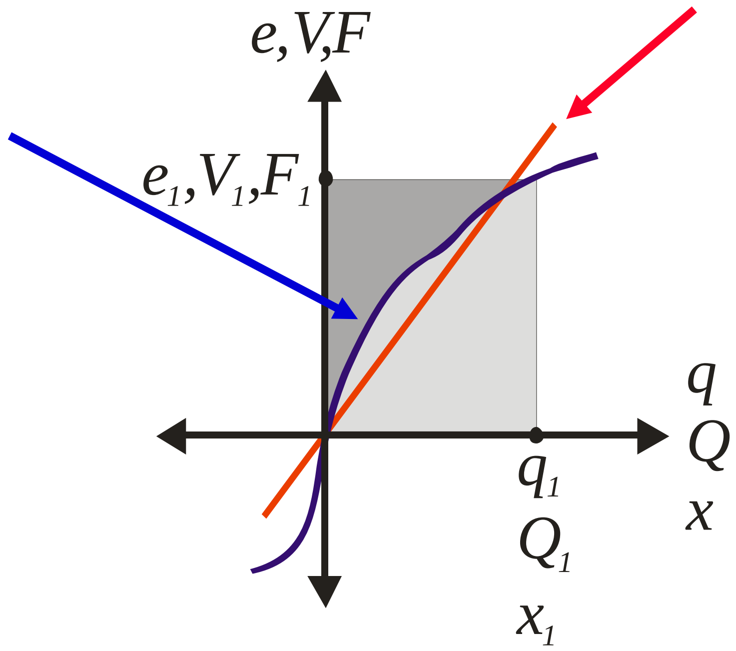
> *Mechanical Power = Fv*

Generalized capacitance

- For a generalized capacitance, the effort e is a function of the generalized displacement q.

Nonlinear capacitor

$$e = \Phi(q)$$



Linear capacitor

$$V = \frac{1}{C} Q = \Phi(Q)$$

$$\Phi(Q) = \frac{Q}{C}$$

Generalized capacitance

> Capacitors store potential energy → How much?

> Leads to concept of energy and co-energy

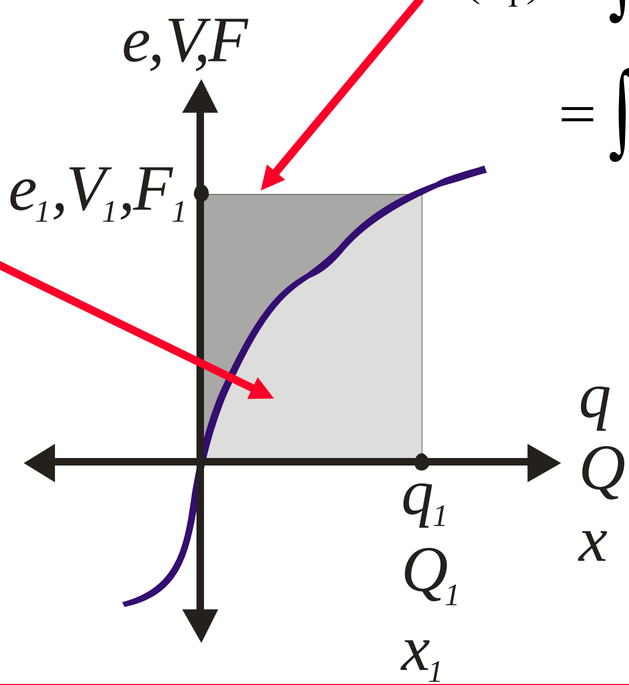
Energy

$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-energy

$$W^*(e_1) = \int_0^{e_1} q\,de$$

$$= \int_0^{e_1} \Phi^{-1}(e)\,de$$



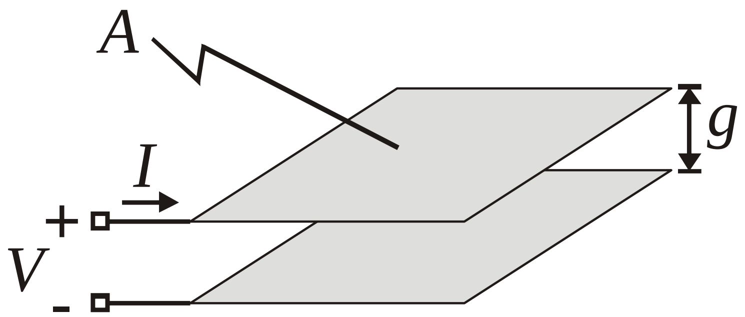
$$\boldsymbol{W(q_1) + W^*(e_1) = e_1 q_1}$$

$$\Downarrow$$

$$\boldsymbol{W^*(e_1) = e_1 q_1 - W(q_1)}$$

Parallel-plate capacitor

- A linear parallel-plate capacitor
- It's energy and co-energy are numerically equal



$$C = \frac{\varepsilon A}{g}$$

$$V = \Phi(Q) = Q/C \qquad Q = \Phi^{-1}(V) = CV$$

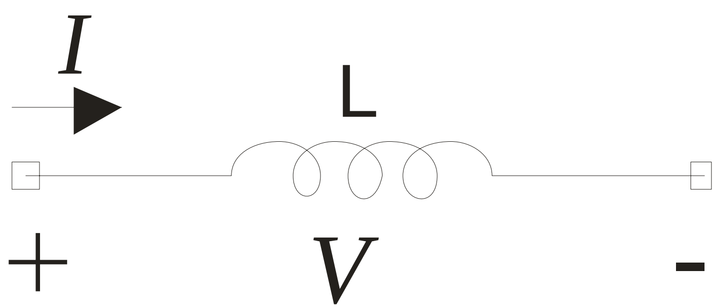
$$W(Q) = \int_0^{Q_1} \Phi(Q)\,dQ = \int_0^{Q_1} Q/C\,dQ \qquad W^*(V) = \int_0^{V_1} \Phi^{-1}(V)\,dV = \int_0^{V_1} CV\,dV$$

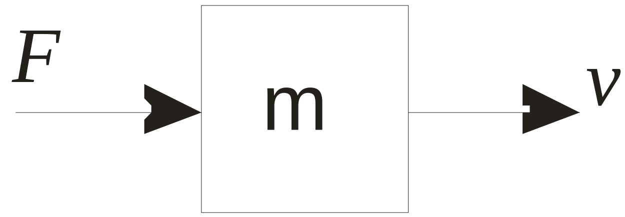
$$W(Q) = \frac{Q^2}{2C} \qquad W^*(V) = \frac{CV^2}{2}$$

Analogies between mechanics and electronics

> **Electrical Domain**
> - **An inductor**

> **Mechanical Domain**
> - **A mass**





$$V = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2} \qquad \boxed{L = m} \qquad F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

> **There is a correspondence between**
> - ***V* and *F***
> - ***I* and *v***
> - ***Q* and *x***
> - ***L* and *m***

> *Electrical Power = VI*

> *Mechanical Power = Fv*

Generalized Inertance

- For a generalized inertance, flow f is a function of momentum p.
- This once again leads to concepts of energy and co-energy

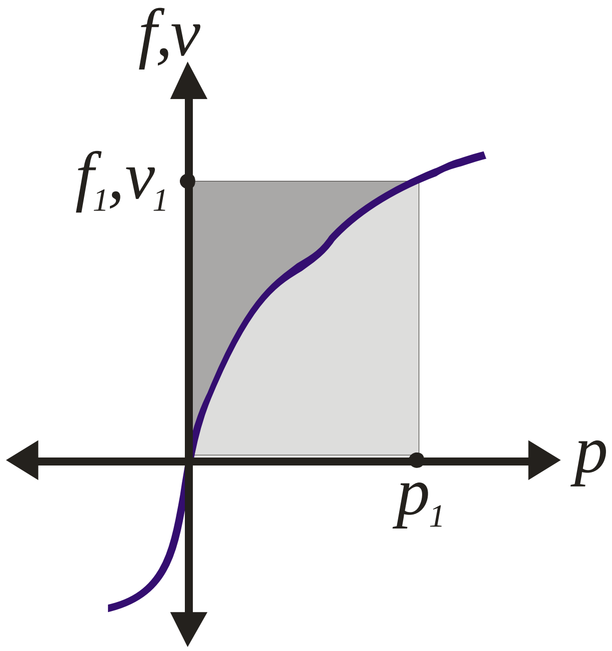
$$W(p_1) = \int_0^{p_1} f dp$$

$$W^*(f_1) = \int_0^{f_1} p df$$

$$W(p_1) + W^*(f_1) = f_1 p_1$$

$$\Downarrow$$

$$W^*(f_1) = f_1 p_1 - W(p_1)$$



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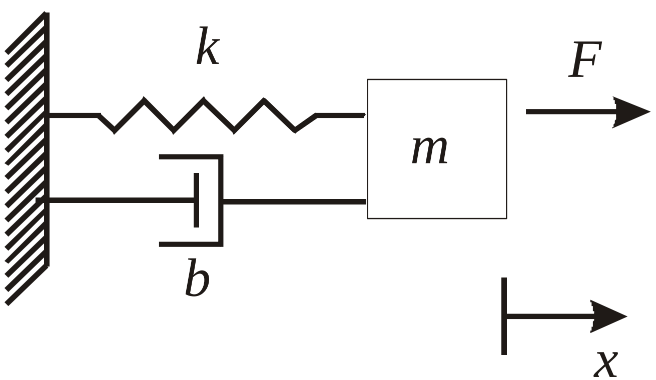
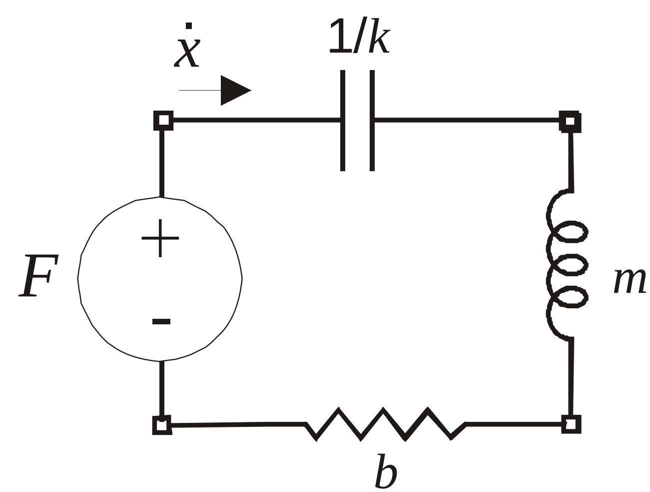
Circuits in the e → V convention

- Elements that share flow (e.g., current) and displacement (e.g., charge) are placed in series in an electric circuit
- Elements that share a common effort (e.g., Voltage) are placed in parallel in an electric circuit



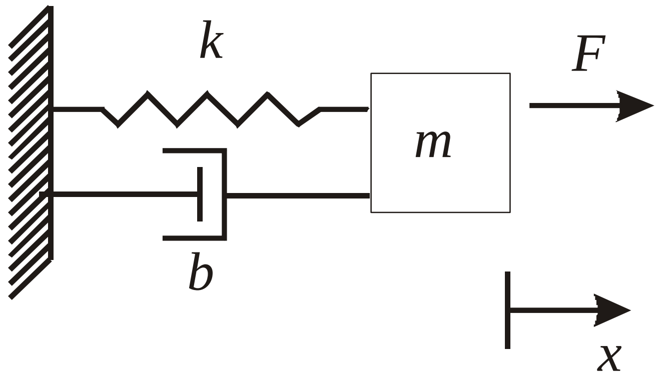
Spring-mass-dashpot system



Equivalent circuit

Solving circuit solves the physics

> **Apply force balance to spring-mass-damper system**



$$\sum_i F_i = ma$$

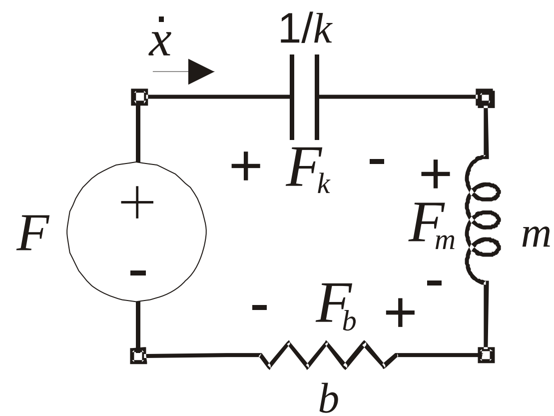
$$F - kx - b\dot{x} = m\ddot{x}$$

> **Solving KVL gives same result as Newton's laws!**

$$F - F_k - F_m - F_b = 0$$

$$F_k = kx, F_b = b\dot{x}, F_m = m\ddot{x}$$

$$F = kx + b\dot{x} + m\ddot{x}$$



> **Can also do this with complex impedances**

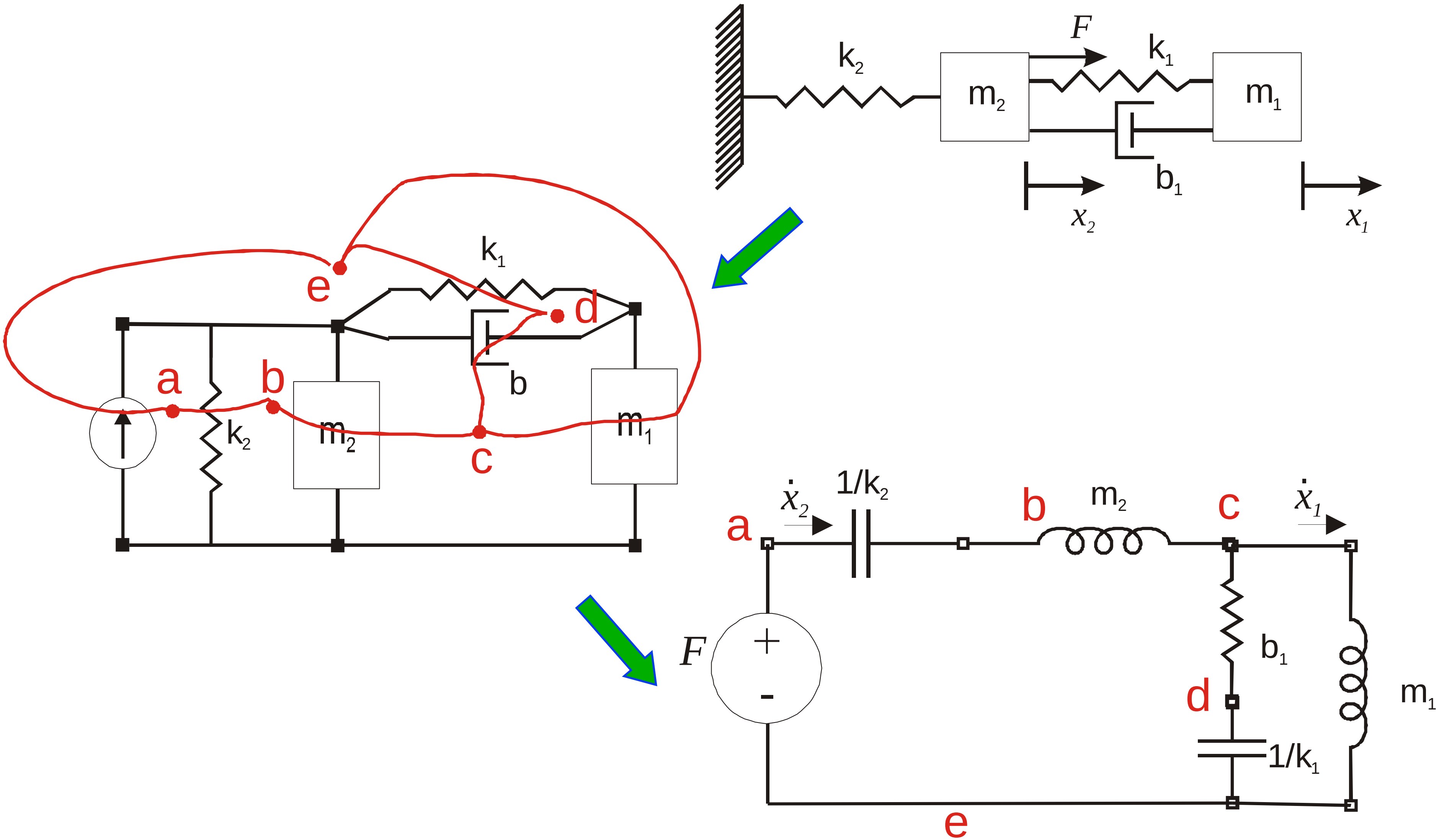
Generating equivalent circuits

> **Possible to go "directly"**
- **But hard with e→V analogy**
- **See slide at end and text for details**

> **Easier to do via circuit duals**

> **Use convenience of *f→V* convention, then switch to *e→V***
- **Force is current source**
- **Each displacement variable is a node**
- **Masses connected between nodes and ground**
- **Other elements connected as shown in diagram**

Example



Where does this leave us?

- A 2nd-order system is a 2nd-order system
- Analogies between RLC and SMD system

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mathbf{m}\,{}^{1}\!/\!_{\mathbf{k}}}} = \sqrt{\frac{\mathbf{k}}{\mathbf{m}}}$$

- Use what you already know to understand the intricacies of what you don’t know

Energy coupling

- Where is coupling between domains?
- How does voltage → deflection?
- We need transducers → two-port elements that store energy
- We will do this next time…

Conclusion

- Can model complicated systems with lumped elements
- Lumped elements from different domains have equivalent-circuit representations
- These representations are not unique
  - We use the e → V convention in assigning voltage to the effort variable
- Once we have circuits, we have access to POWERFUL analysis tools

For more info

- Course text chapter 5
- H.A.C. Tilmans. "Equivalent circuit representation of electromechanical transducers"
  - Part I: lumped elements: *J. Micromech. Microeng.* 6:157, 1996.
  - Part II: distributed systems: *J. Micromech. Microeng.* 7:285, 1997.
  - Errata: *J. Micromech. Microeng.* 6:359, 1996.
- R. A. Johnson. *Mechanical filters in electronics*
- Woodson and Melcher. *Electromechanical Dynamics*
- M. Rossi. *Acoustics and electroacoustics*
- Lots and lots of papers

Finding equivalent circuit: direct approach

- **Find e→V equivalent circuit of following**
- **Note:**
  - **k_2 and m_2 share same displacement, caused by F**
  - **b_1, and k_1 share same displacement, $x_2 - x_1$**
  - **If $k_1 \to \infty$, m_2 and m_1 share same displacement**