6.730 Physics for Solid State Applications

Lecture 23: Effective Mass

Outline

- Review of Last Time
- A Closer Look at Valence Bands
- k.p and Effective Mass

Semiclassical Equations of Motion

$$< \mathbf{v_n(k)} >= \frac{< \mathbf{p} >}{\mathbf{m}} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \mathbf{E_n(k)}$$

$$\mathbf{F_{ext}} = \hbar \frac{\mathbf{dk}}{\mathbf{dt}}$$

Lets try to put these equations together….

$$a(t) = \frac{dv}{dt} = \frac{1}{\hbar}\frac{\partial}{\partial t}\frac{\partial E_N(k)}{\partial k} = \frac{1}{\hbar}\frac{\partial^2 E_N(k)}{\partial k^2}\frac{dk}{dt}$$

$$= \left[\frac{1}{\hbar^2}\frac{\partial^2 E_N(k)}{\partial k^2}\right] F_{\text{ext}}$$

Looks like Newton's Law if we define the mass as follows…

$$m^*(k) = \hbar^2 \left(\frac{\partial^2 E_N(k)}{\partial k^2}\right)^{-1}$$

dynamical effective mass

→ mass changes with k…so it changes with time according to k

Dynamical Effective Mass (3D)

Extension to 3-D requires some care,

F and a don't necessarily point in the same direction

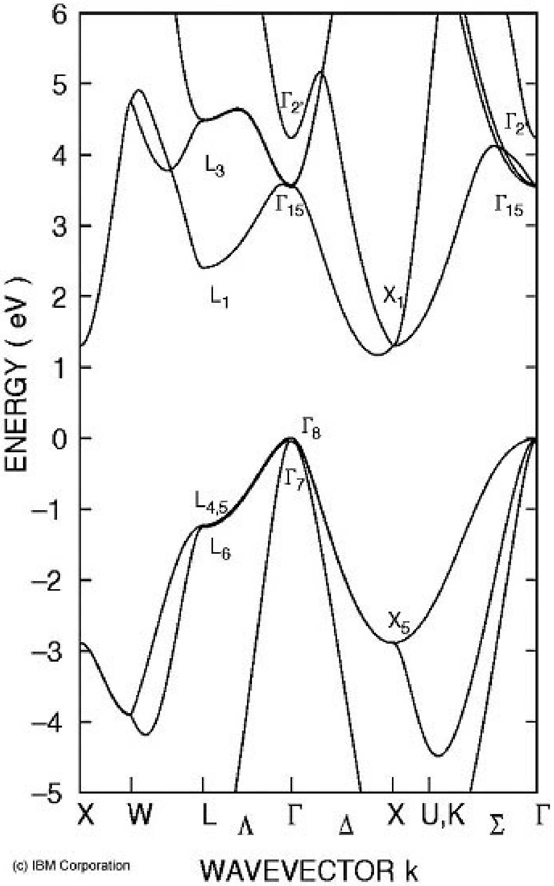
$$\mathbf{a} = \overline{\overline{\mathbf{M}}}^{-1}\mathbf{F}_{\text{ext}} \quad \text{where} \quad \overline{\overline{\mathbf{M}}}^{-1}_{i;j} = \frac{1}{\hbar^2}\frac{\partial^2 E_N}{\partial k_i \partial k_j}$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \frac{1}{m_{xx}} & \frac{1}{m_{xy}} & \frac{1}{m_{xz}} \\ \frac{1}{m_{yx}} & \frac{1}{m_{yy}} & \frac{1}{m_{yz}} \\ \frac{1}{m_{zx}} & \frac{1}{m_{zy}} & \frac{1}{m_{zz}} \end{pmatrix}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

Dynamical Effective Mass (3D)
Ellipsoidal Energy Surfaces

Fortunately, energy surfaces can often be approximate as…

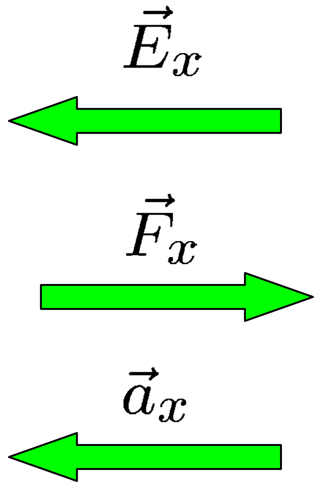
$$E_N(k) = E_c + \frac{\hbar^2}{2}\left(\frac{(k_x - k_x^0)^2}{m_t} + \frac{(k_y - k_y^0)^2}{m_t} + \frac{(k_z - k_z^0)^2}{m_l}\right)$$



$$\overline{\overline{\mathbf{M}}}^{-1} = \begin{pmatrix} \frac{1}{m_t} & 0 & 0 \\ 0 & \frac{1}{m_t} & 0 \\ 0 & 0 & \frac{1}{m_l} \end{pmatrix}$$

$$\overline{\overline{\mathbf{M}}} = \begin{pmatrix} m_t & 0 & 0 \\ 0 & m_t & 0 \\ 0 & 0 & m_l \end{pmatrix}$$

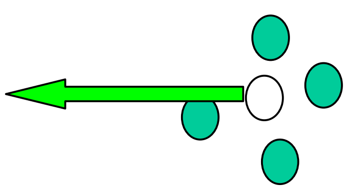
Motion of Valence Electrons (and Holes)



electrons have negative charge

valence electrons have negative mass !

<u>Real space</u>



Vacancy ends up moving in the direction of the electric field as if it had a positive charge

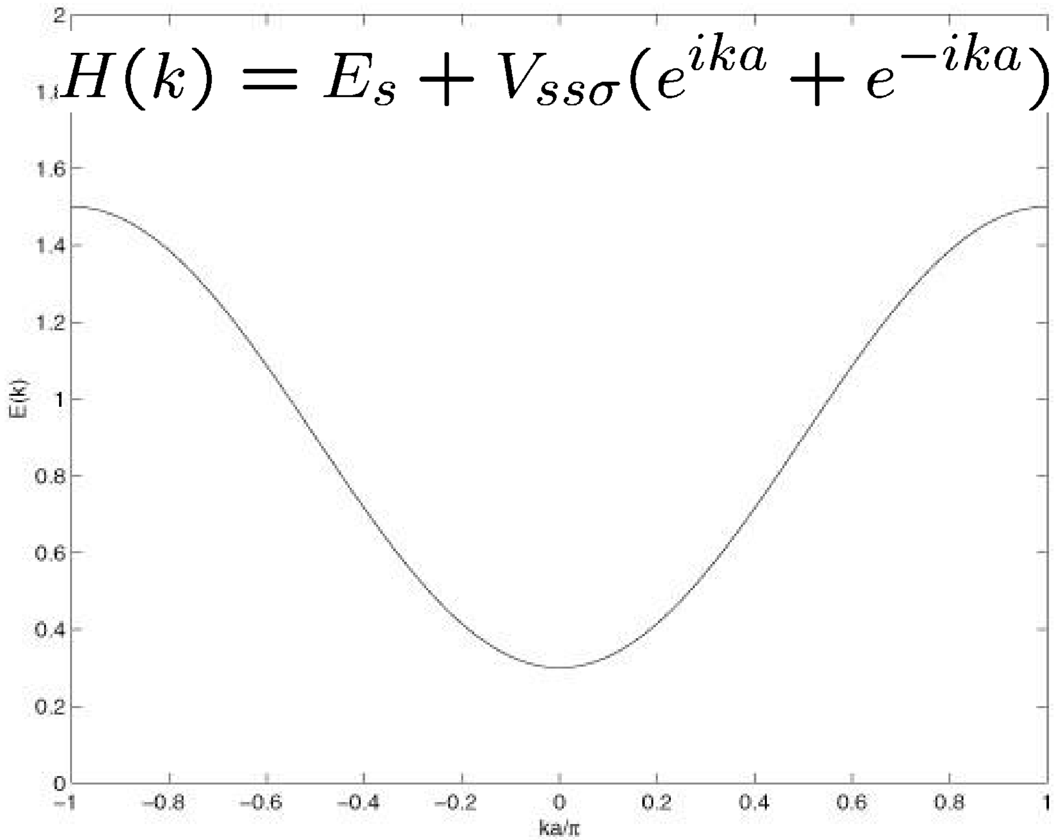
Hole is a quasi-particle with positive charge and positive mass…

Energy Band for 1-D Lattice

Single orbital, single atom basis

$$m^*(k) = \hbar^2 \left(\frac{\partial^2 E_N(k)}{\partial k^2} \right)^{-1}$$

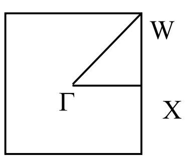
$$H(k) = E_s + V_{ss\sigma}(e^{ika} + e^{-ika})$$

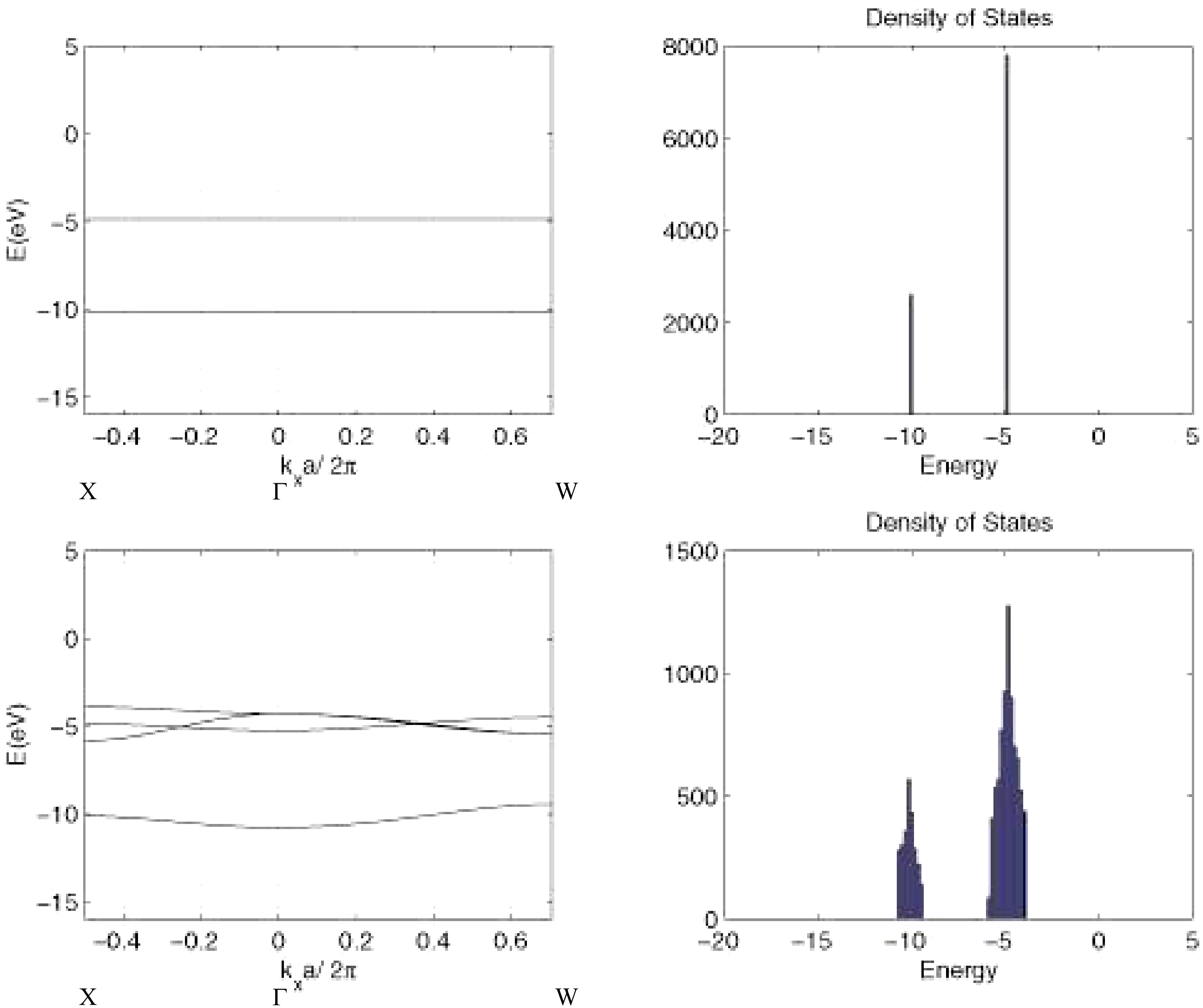


Increasing the orbital overlap, reduces the effective mass…

2D Monatomic Square Crystals

Variations with Lattice Constant





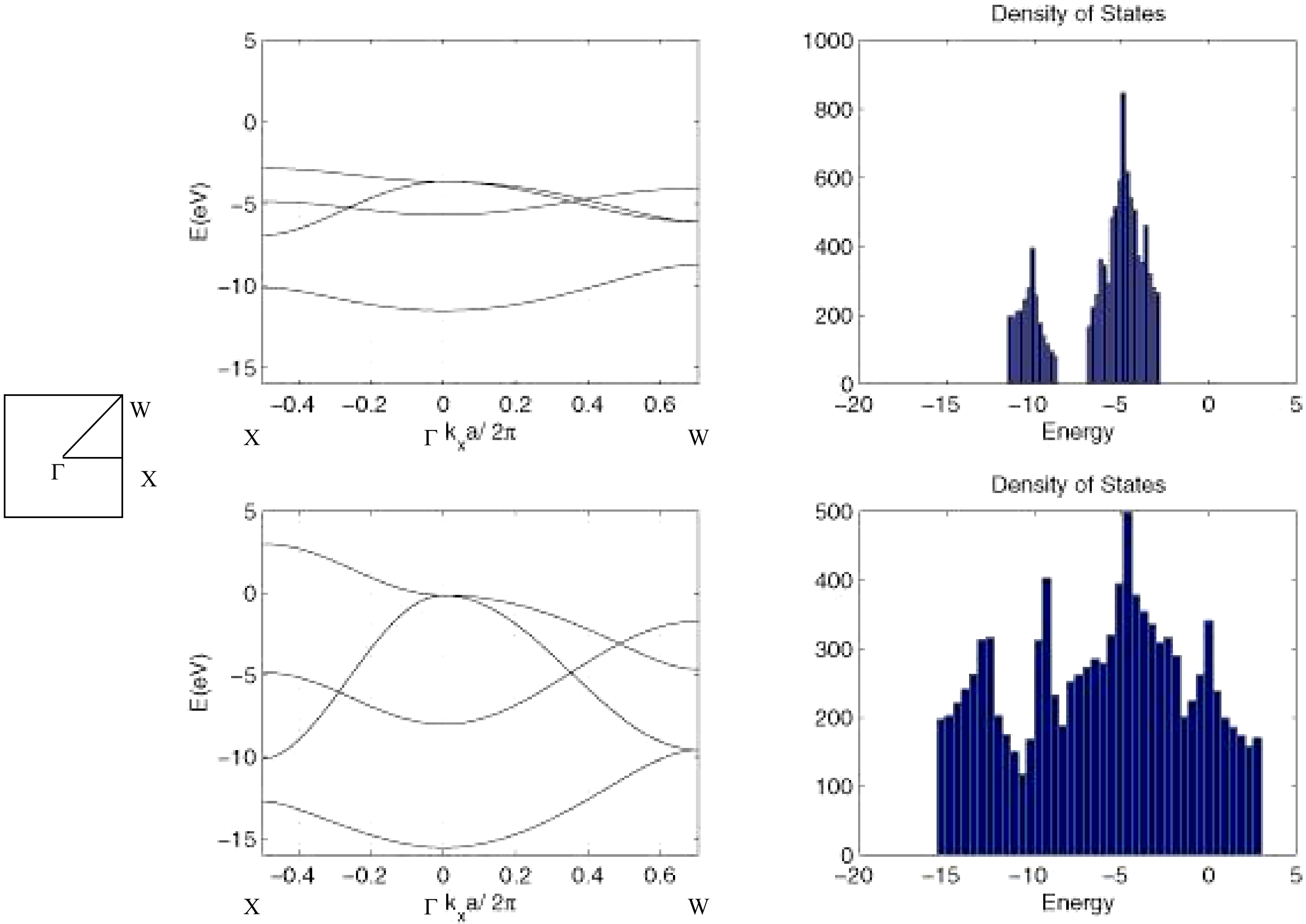
$a \rightarrow \infty$

$a = 8.3\ \text{A}$

Increasing the orbital overlap, reduces the effective mass…

2D Monatomic Square Crystals

Dispersion Relations



$a = 5.5$ A

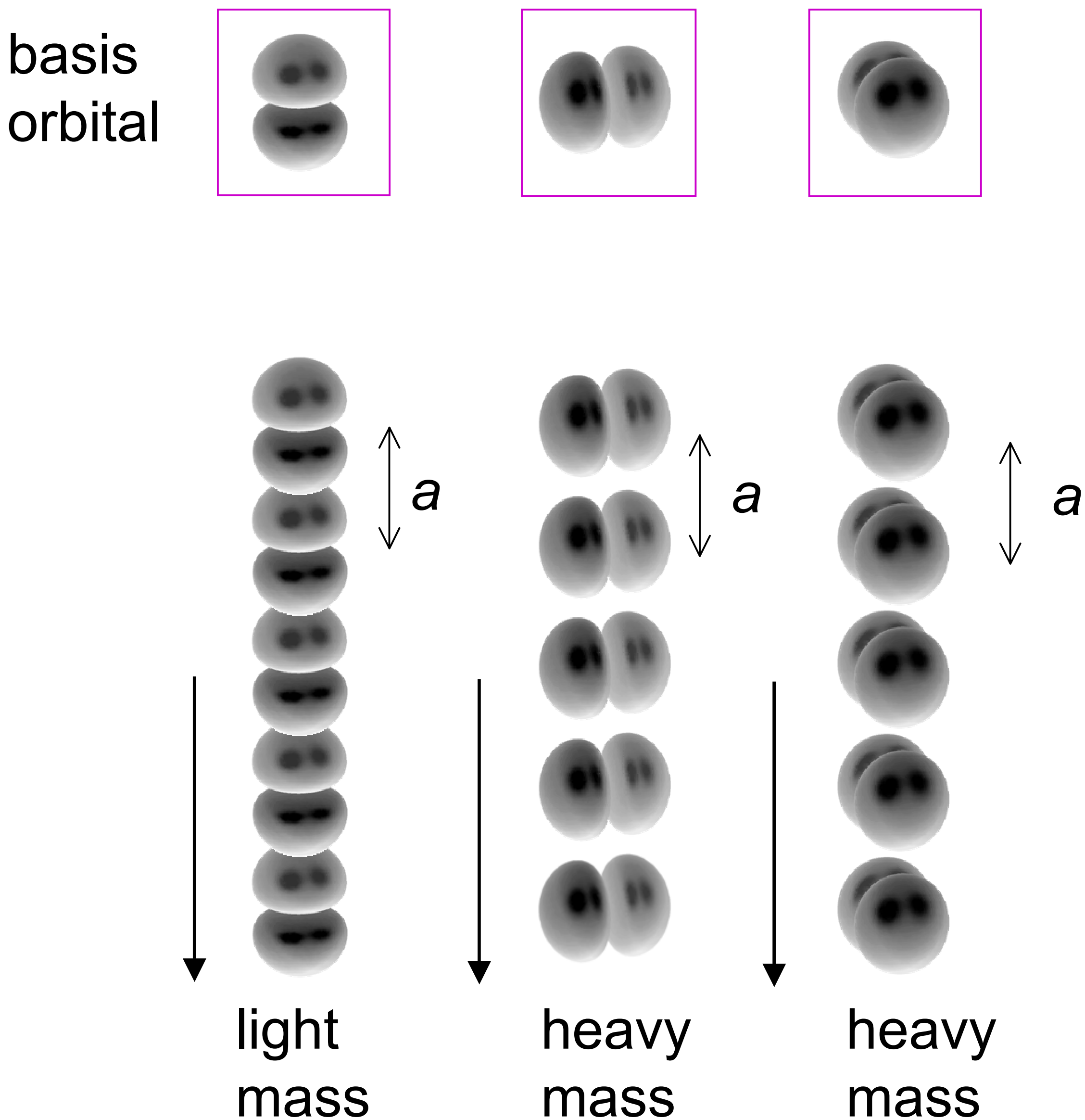
$a = 2.8$ A

Increasing the orbital overlap, reduces the effective mass…

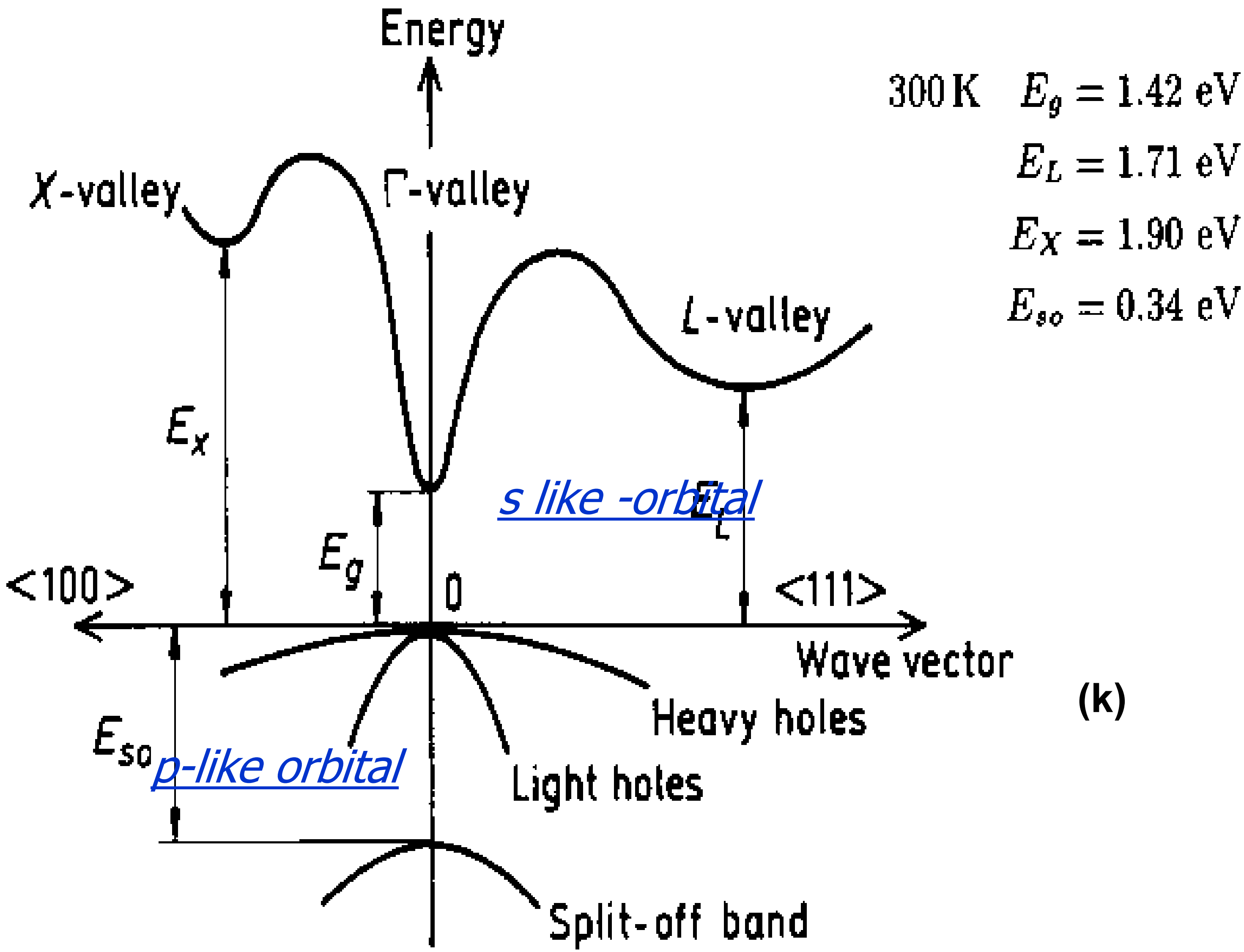
3D Band Structures

Dispersion Relations

Lighter effective mass ⟷ Larger overlap between orbitals



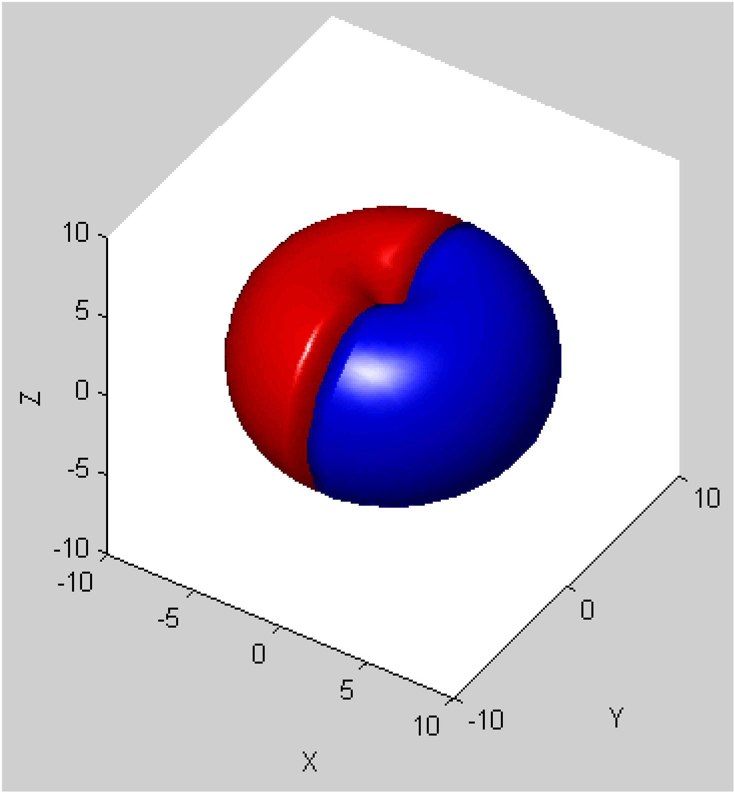
Bandstructure of GaAs



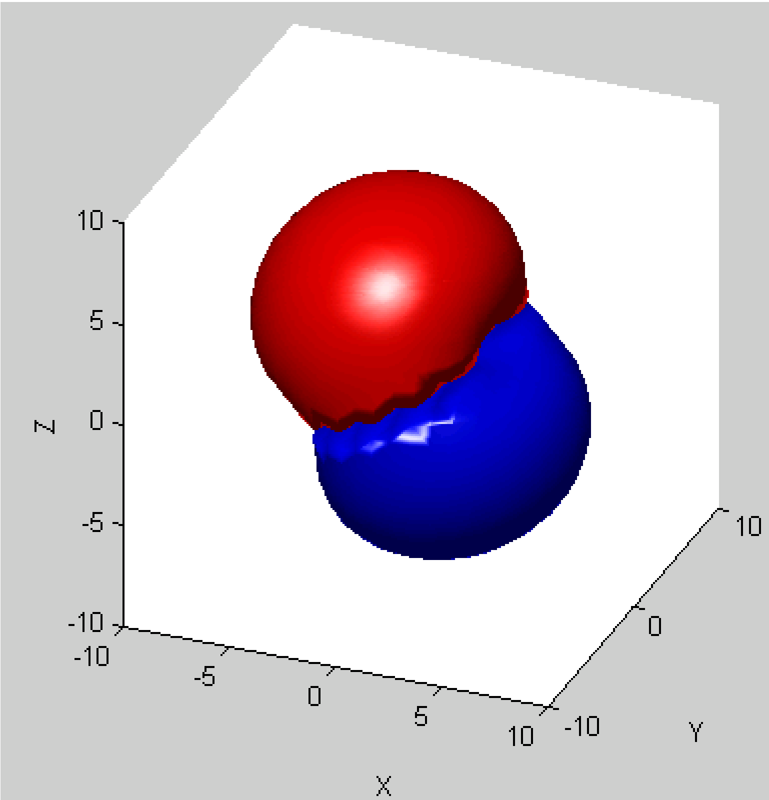
What is this split-off band ?

Spin-orbit Coupling Wavefunctions

heavy hole charge distribution



light hole charge distribution



Orbital Angular Momentum

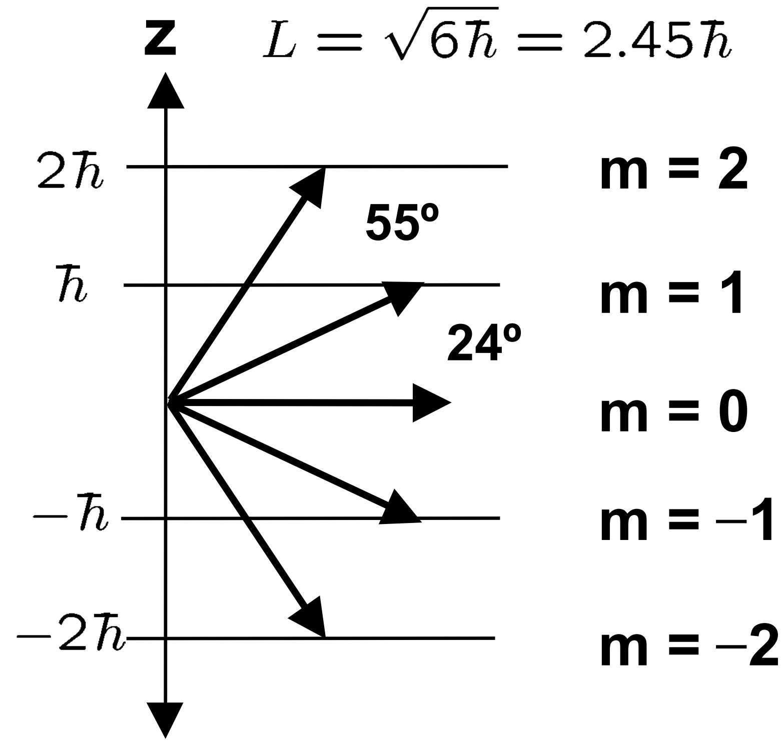
Angular momentum for quantum state with *l* = 2:

$l = 2$

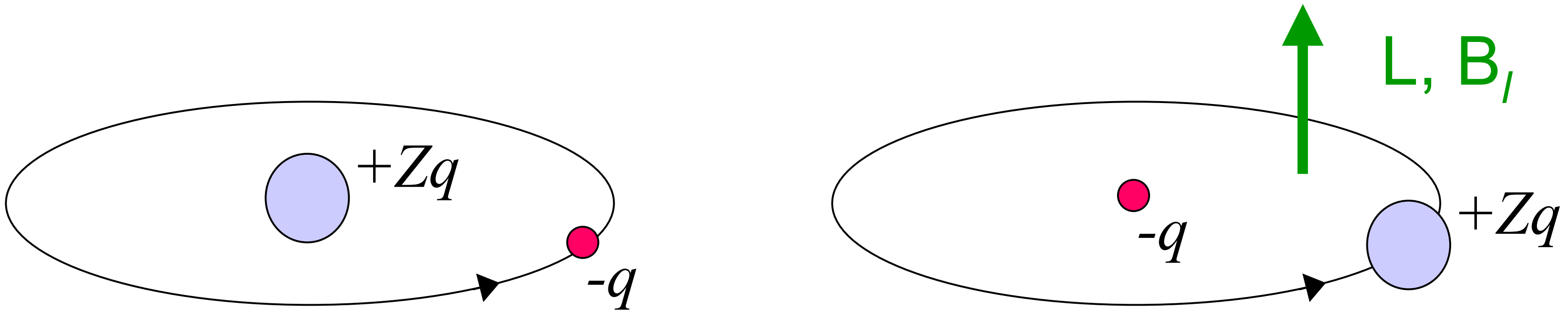
$$L = \sqrt{l(l+1}\hbar = \sqrt{6}\hbar = 2.45\hbar$$

$$m = -l \text{ to l} = 0, \pm 1, \pm 2$$

$$L_z = 0, \pm\hbar, \pm 2\hbar$$



Spin-Orbit Coupling



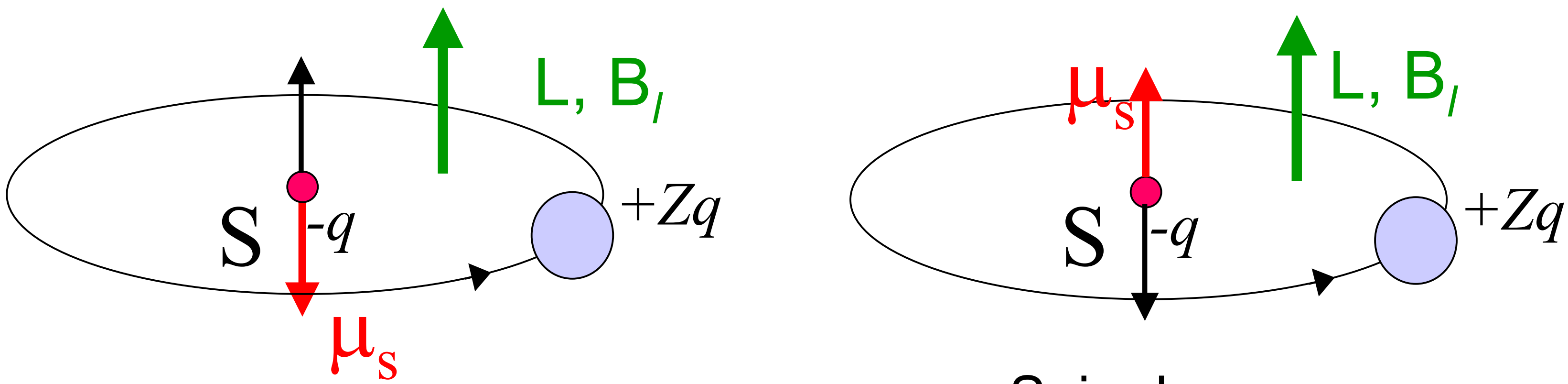
The effective current from the motion of a nucleus in a circular orbit…

$$I = \frac{\Delta Q}{\Delta t} = \frac{Zev}{2\pi r}$$

…generates an effective magnetic field…

$$B = \frac{\mu_0 I}{2r} \Rightarrow B = \frac{\mu_0 Zev}{4\pi r^2}$$

Spin-Orbit Splitting

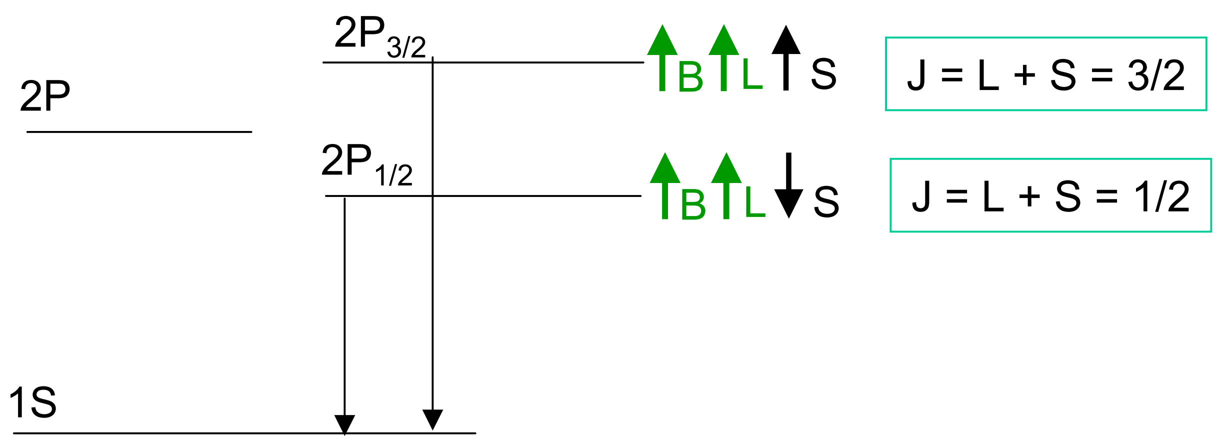


<u>Spin up:</u>

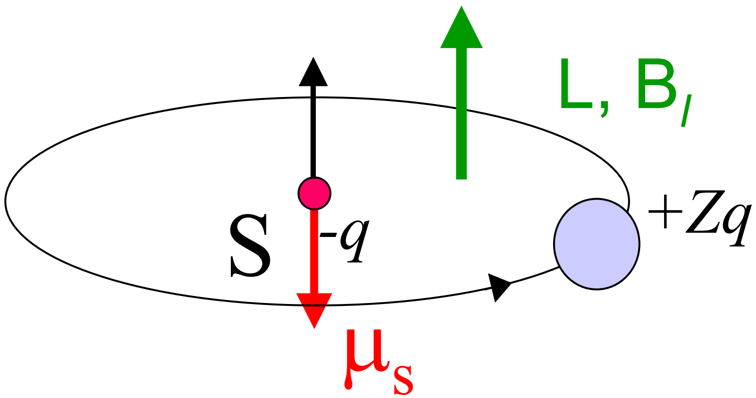
High Energy

<u>Spin down:</u>

Low Energy

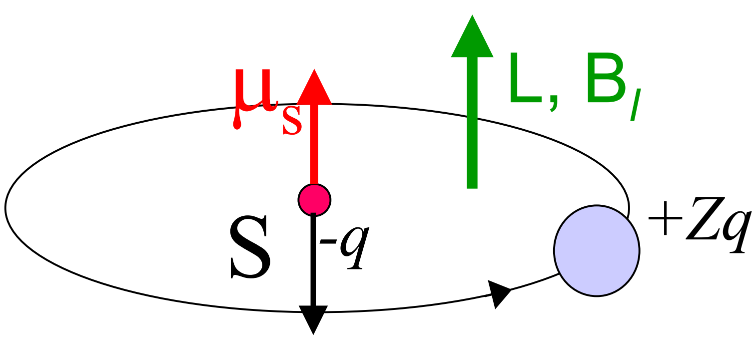


Spin-Orbit Splitting in Hydrogen



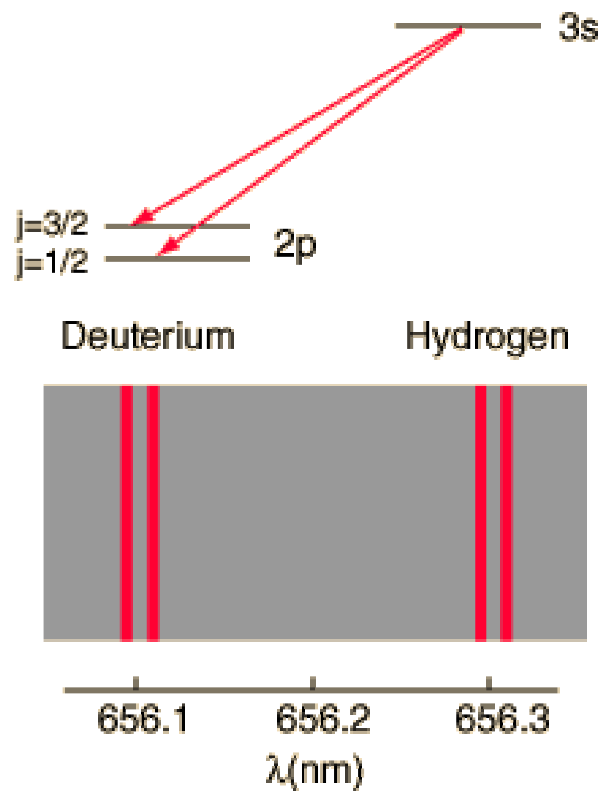
Spin up:

High Energy



Spin down:

Low Energy



Angular Momentum Addition Rules

Vectors

$$J = L + S$$

$$|J| = \sqrt{j(j+1)}\hbar$$

Quantum Numbers

$$j = l + s, |l - s|$$

$$m_j = -j, -j+1, \ldots j-1, j$$

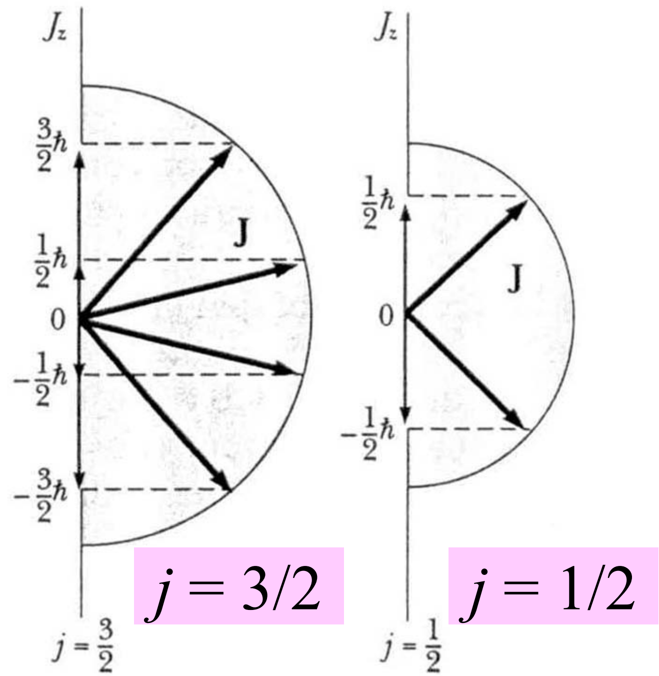
Example: $l = 1, s = ½$

$$j = 1 + \frac{1}{2} = \frac{3}{2}$$

$$m_j = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$$

$$j = |1 - \frac{1}{2}| = \frac{1}{2}$$

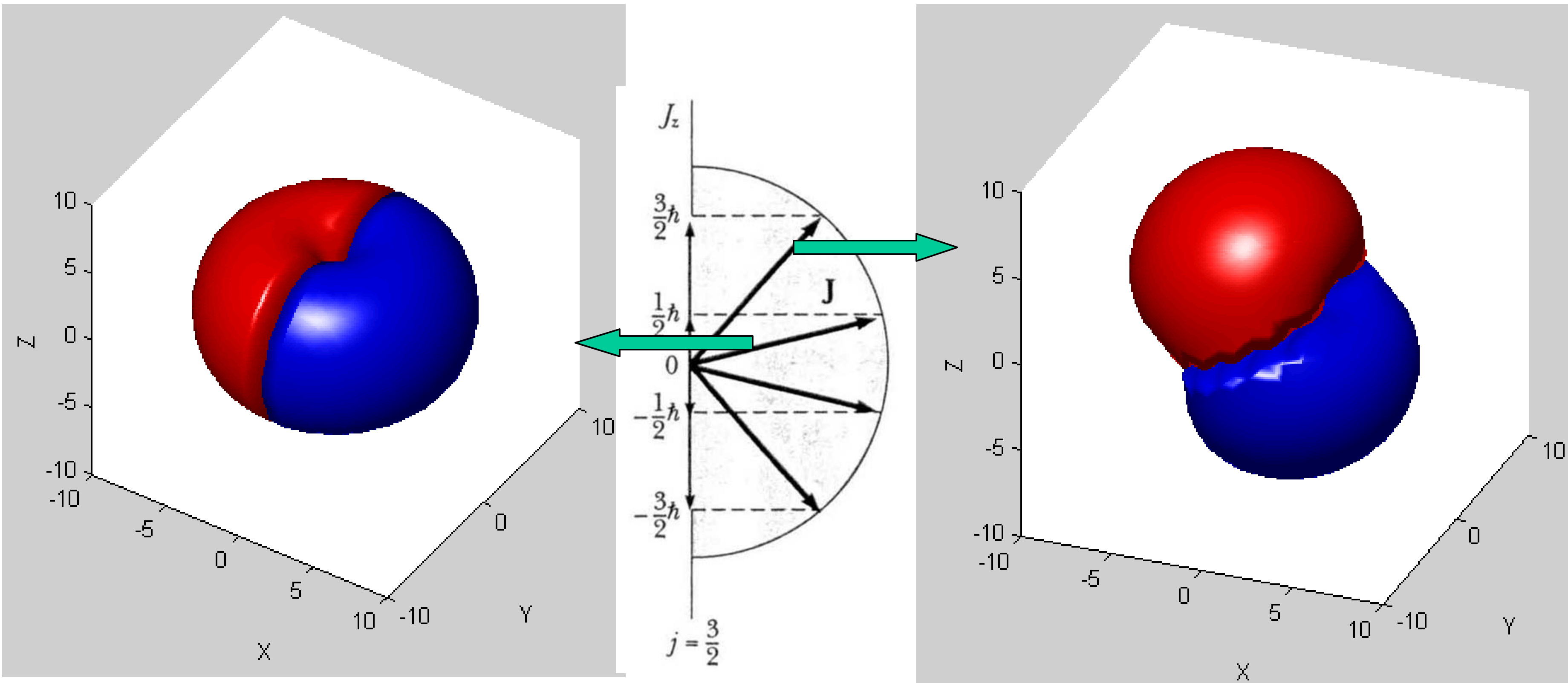
$$m_j = -\frac{1}{2}, +\frac{1}{2}$$



Spin-orbit Coupling Wavefunctions

heavy hole charge distribution

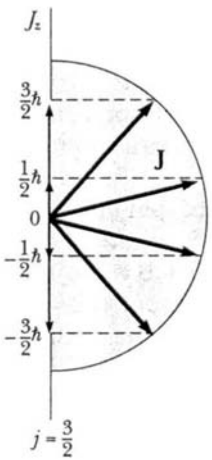
light hole charge distribution



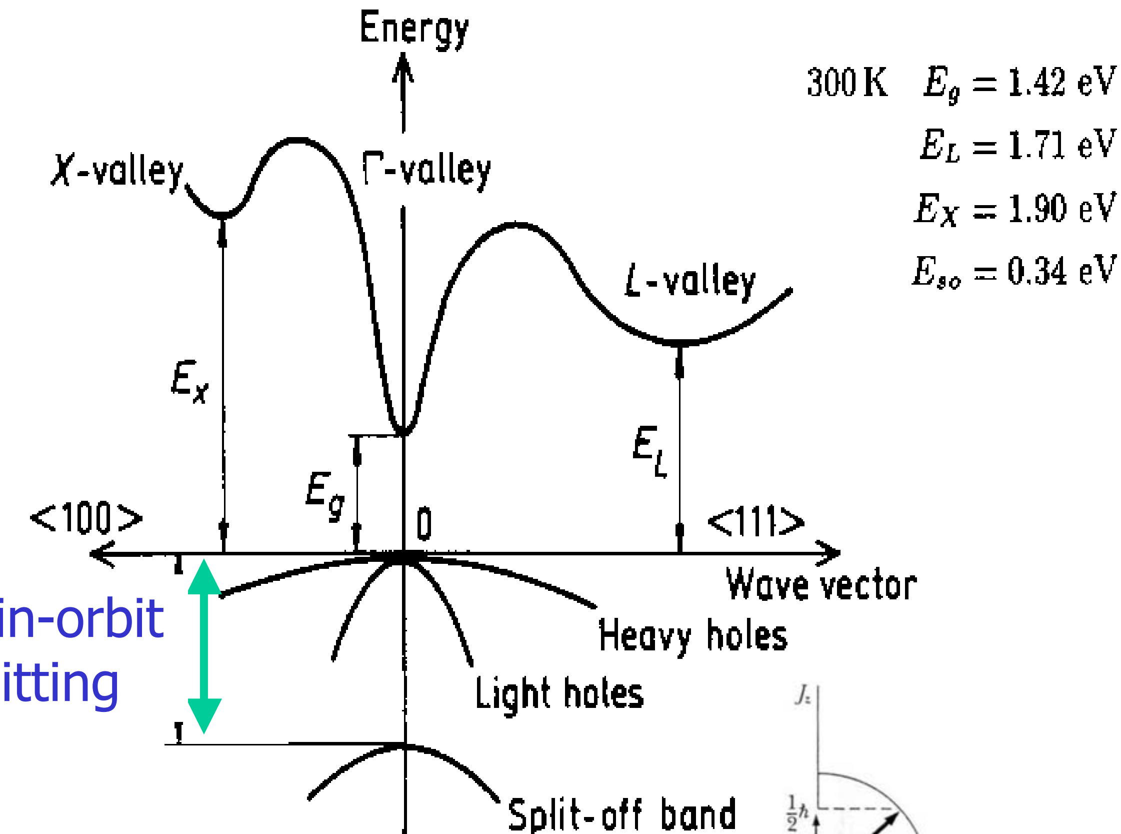
heavy mass (along k_z)

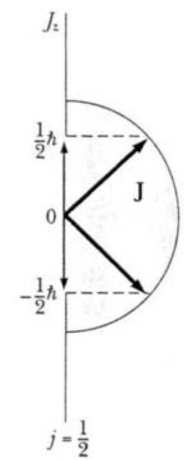
light mass (along k_z)

Bandstructure of GaAs

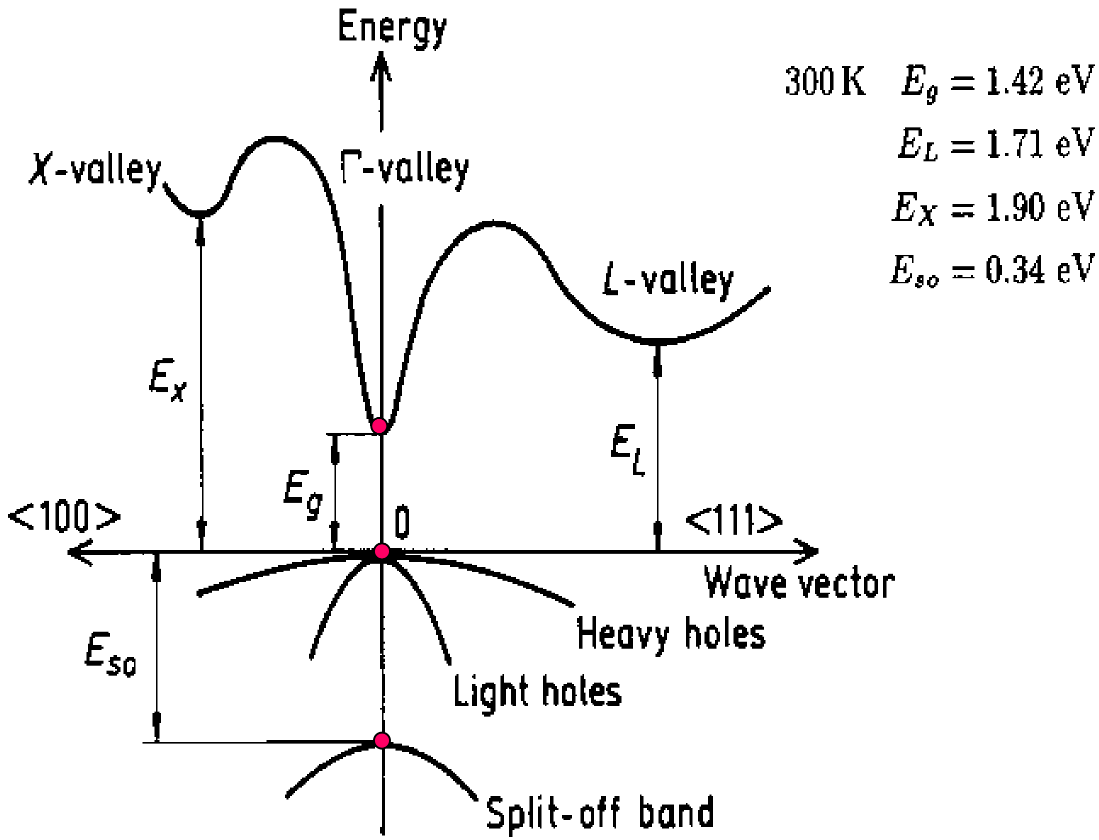


Spin-orbit splitting





Another Approach to Bandstructure: k.p



Often it is easier to know the energies at a particular point (ex. Bandgap) than it is to measure the effecitve mass

k.p is a way to relate your knowledge of energy levels at k to the effective mass…using perturbation theory

Momentum and Crystal Momentum

$$\hat{\mathbf{p}}\,\psi_{n,k} = \hbar k \psi_{n,k} + e^{ik\cdot r}\frac{\hbar}{i}\nabla \tilde{u}_{n,k}(r)$$

$$\hat{\mathbf{p}}\,\psi_{n,k} = e^{ik\cdot r}\hbar\left(k + \frac{1}{i}\nabla\right)\tilde{u}_{n,k}(r)$$

Leads us to, the action of the Hamiltonian on the Bloch amplitude….

$$e^{ik\cdot r}\left(\frac{\hbar^2}{2m}\left(\frac{1}{i}\nabla + k\right)^2 + V(r)\right)\tilde{u}_k(r) = E_k e^{ik\cdot r}\tilde{u}_k(r)$$

$$H_k \tilde{u}_k(r) \equiv \left(\frac{\hbar^2}{2m}\left(\frac{1}{i}\nabla + k\right)^2 + V(r)\right)\tilde{u}_k(r) = E_k \tilde{u}_k(r)$$

k.p Hamiltonian
(in our case q.p)

$$H_k \tilde{u}_k(r) = \left(\frac{\hbar^2}{2m} \left(\frac{1}{i} \nabla + k \right)^2 + V(r) \right) \tilde{u}_k(r)$$

If we know energies as k we can extend this to calculate energies at k+q for small q…

$$H_{k+q} = \frac{\hbar^2}{2m} \left(\frac{1}{i} \nabla + k + q \right)^2 + V(r)$$

$$H_{k+q} = H_k + \underbrace{\frac{\hbar^2}{m} q \cdot \left(\frac{1}{i} \nabla + k \right) + \frac{\hbar^2}{2m} q^2}_{\text{perturbation}}$$

k.p Effective Mass

$$H_{k+q} = H_k + \underbrace{\frac{\hbar^2}{m} q \cdot \left(\frac{1}{i}\nabla + k\right) + \frac{\hbar^2}{2m} q^2}_{\text{perturbation(V)}}$$

Second-order perturbation theory…

$$E_n^{(2)} \approx E_n^0 + V_{nn} + \sum_{p \neq n} \frac{|V_{np}|^2}{E_n^0 - E_p^0} \qquad \text{provided} \qquad E_n^0 \neq E_p^0$$

Taylor Series expansion of energies…

$$E_n(k+q) = E_n(k) + \sum_i \frac{\partial E_n}{\partial k_i} q_i + \frac{1}{2} \sum_{ij} \underbrace{\frac{\partial^2 E_n}{\partial k_i \partial k_j}}_{\vec{\vec{\mathrm{M}}}_{\mathrm{i,j}}} q_i q_j + O(q^3)$$

$$\sum_{ij} \frac{1}{2} \frac{\partial^2 E_n}{\partial k_i \partial k_j} q_i q_j = \frac{\hbar^2}{2m} q^2 + \sum_{n' \neq n} \frac{|<\frac{\hbar^2}{2m} q \cdot \left(\frac{1}{i}\nabla + k\right)>|^2}{E_{nk} - E_{n'k}}$$

k.p Effective Mass

$$\sum_{ij} \frac{1}{2} \frac{\partial^2 E_n}{\partial k_i \partial k_j} q_i q_j = \frac{\hbar^2}{2m} q^2 + \sum_{n' \neq n} \frac{|< \frac{\hbar^2}{2m} q \cdot \left(\frac{1}{i} \nabla + k \right) > |^2}{E_{nk} - E_{n'k}}$$

$$= \frac{\hbar^2}{2m} q^2 + \sum_{n' \neq n} \frac{| \int dr\, \tilde{u}^*_{nk} \frac{\hbar^2}{2m} q \cdot \left(\frac{1}{i} \nabla + k \right) \tilde{u}_{n'k} |^2}{E_{nk} - E_{n'k}}$$

$$= \frac{\hbar^2}{2m} q^2 + \sum_{n' \neq n} \frac{| < \psi_{nk} | \frac{\hbar^2}{2m} q \cdot \frac{1}{i} \nabla | \psi_{n'k} > |^2}{E_{nk} - E_{n'k}}$$

$$= \frac{\hbar^2}{2m} q^2 + \left(\frac{\hbar^2}{m} \right)^2 \sum_{n' \neq n} \frac{| < q \cdot \hat{p} >_{nn'} |^2}{E_{nk} - E_{n'k}}$$

k.p Effective Mass
Example

$$\frac{\partial^2 E_n}{\partial k_i \partial k_j} = \frac{\hbar^2}{m}\delta_{i,j} + \left(\frac{\hbar^2}{m}\right)^2 \sum_{n' \neq n} \frac{< \hat{p}_i >_{nn'} < \hat{p}_j >_{n'n} + < \hat{p}_i >_{n'n} < \hat{p}_j >_{nn'}}{E_{nk} - E_{n'k}}$$

$$\overline{\overline{\mathbf{M}}}^{-1}_{i;j} = \frac{1}{\hbar^2}\frac{\partial^2 E_N}{\partial k_i \partial k_j}$$

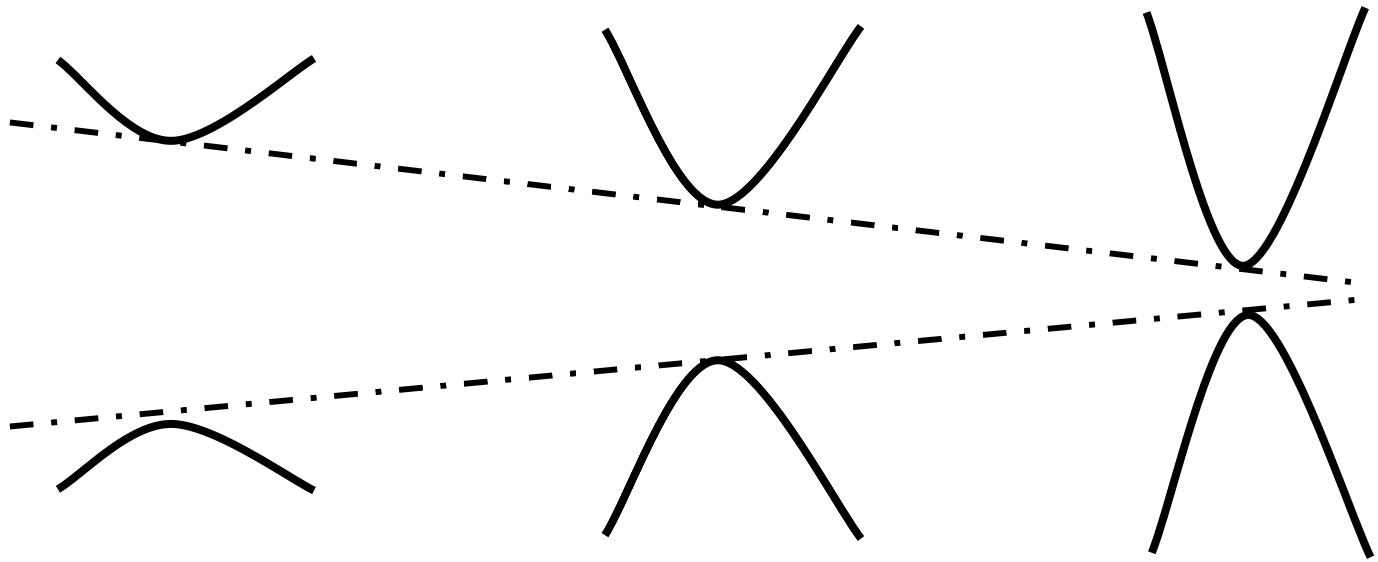
Lets only consider two bands (valence and conduction) and assume they are spherical…

$$\frac{1}{m^*} = \frac{1}{m} + 2\left(\frac{\hbar^2}{m^2}\right)\frac{| < \hat{p} >_{cv} |^2}{E_{c0} - E_{v0}}$$

$$= \frac{1}{m} + 2\left(\frac{\hbar^2}{m^2}\right)\frac{| < \hat{p} >_{cv} |^2}{E_g}$$

k.p Effective Mass

Example



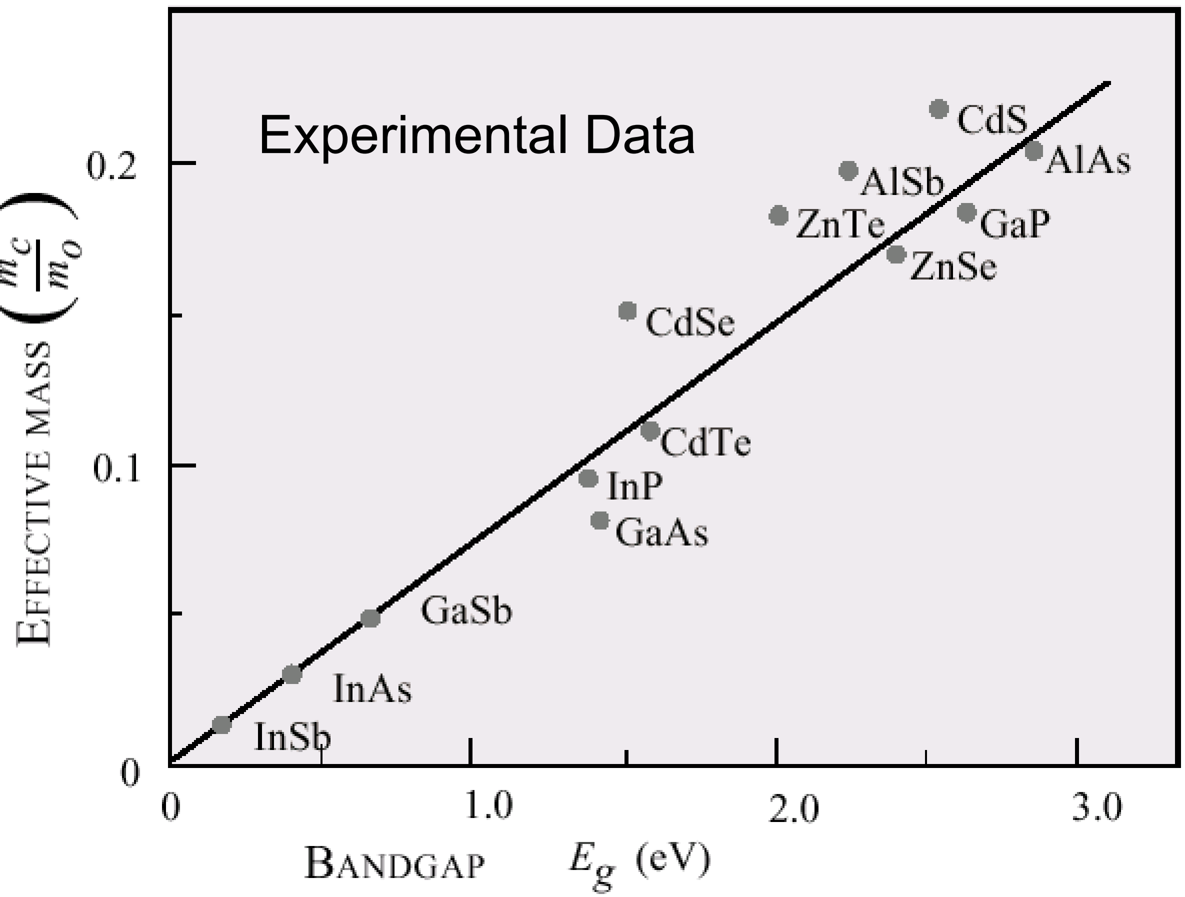
$$\frac{1}{m^*} = \frac{1}{m} + 2\left(\frac{\hbar^2}{m^2}\right)\frac{|<\hat{p}>_{cv}|^2}{E_g}$$

$$E_n^{(2)} \approx E_n^0 + V_{nn} + \sum_{p\neq n}\frac{|V_{np}|^2}{E_n^0 - E_p^0}$$

Level repulsion causes bands to curve as bandgap is reduceed…

Effective Mass and Bandgap

$$\frac{1}{m^*} = \frac{1}{m} + 2\left(\frac{\hbar^2}{m^2}\right)\frac{|<\hat{p}>_{cv}|^2}{E_g}$$



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(see "Semiconductor Bandstructure")