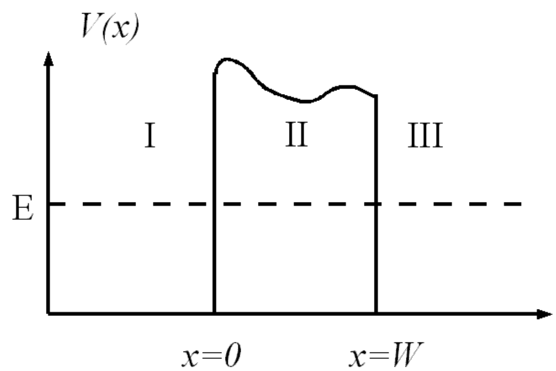
WKB Approximation Applied to Tunneling



$$
\begin{aligned}
\Psi_I &= e^{ikx} + r_I e^{-ikx} \\
\Psi_{II} &= t_{II}(x) e^{-\theta(x)} + r_{II}(x) e^{\theta(x)} \\
k &= \sqrt{\frac{2m}{\hbar^2} E} \\
\theta(x) &= \int_0^x \sqrt{\frac{2m}{\hbar^2} (V(x') - E)}\, dx' \\
\Psi_{III} &= A_{III} e^{ikx}
\end{aligned}
$$

Note, we use $t(x)$ and $r(x)$ in Ψ_{II}, but they can only take the WKB AWAY from the classical turning point.

Now we assume $r_{II} = 0$. This is safe as long as the barrier remains relatively thick (so the reflected wave has small amplitude).

Now consider the coundary condition at $x = 0$:

$$
\Psi_I(0) = t_{II}(0) e^{-\theta(0)}
$$

$$\Rightarrow \Psi_{II}(x) \quad = \quad \Psi_I(0)\frac{e^{-\theta(x)}}{\sqrt{\zeta(x)}}, \qquad \text{where } \zeta(x) = \sqrt{\frac{2m}{\hbar}(V(x)-E)}.$$

Keep in mind that $\Psi_{II}(x)$ is the approximate eigenstate only (e.g at $x = 0$, $\Psi_{II} \to \infty$ which is clearly unphysical).

Now consider the boundary condition for continuity at W:

$$\Psi_{II}(W) \quad = \quad \frac{\Psi_I(0)e^{-\int_0^W \sqrt{\frac{2m}{\hbar^2}(V(x')-W)}\,dx'}}{\sqrt[4]{\frac{2m}{\hbar^2}(V(W)-E)}} = \Psi_{III}(W).$$

Then

$$\begin{aligned} T &= \left|\frac{\Psi_{III}(W)}{\Psi_I(0)}\right|^2 \\ &= \frac{e^{-2\int_0^W \sqrt{\frac{2m}{\hbar^2}(V(x')-E)}\,dx'}}{\sqrt{\frac{2m}{\hbar^2}(V(W)-E)}} \end{aligned}$$

This solution has the form $T = Ae^{-2\theta(x)}$. In general tunneling barriers will have a dependence of

$$T \cong e^{-\sqrt{\phi_0}W},$$

which falls out of the derivation above if $V(x) = \phi_0,\ 0 < x < W$ (so the barrier height is a constant). More complex barriers may require careful corrections to WKB to achieve quantitative agreement with experiments.