6.728 Applied Quantum and Statistical Physics:

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PROBLEM SET 7

PROBLEMS

Problem 7.1   *NOT and $\sqrt{\text{NOT}}$ Logic Operations*
   Consider the two level system. Let the energy between the two levels in the absence coupling be the same so that $\Delta = 0$. Then with the attractive coupling $V = -|V|$, the evolution matrix is (to within an overall phase factor) is given by Eqn. 17.75 in the text:

$$U(t) = \begin{pmatrix} \cos|V|t/\hbar & i\sin|V|t/\hbar \\ i\sin|V|t/\hbar & \cos|V|t/\hbar \end{pmatrix} \tag{1}$$

   The NOT operator is given by

$$\text{NOT} = -iU_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

where U_α is $U(t)$ evaluated at time such that $2Vt/\hbar = \alpha$.

(a) Show that $U_\beta U_\alpha = U_{\alpha+\beta}$ by explicitly multiplying the matrices. What physical reason can you give for this?

(b) If $\text{NOT} = -iU_\pi$, then argue that the operation

$$\sqrt{\text{NOT}} = \frac{1-i}{\sqrt{2}}U_{\pi/2}$$

where $\sqrt{\text{NOT}}\sqrt{\text{NOT}} = \text{NOT}$
(c) Which of the operators, U_α, NOT, and $\sqrt{\text{NOT}}$, are unitary?

Problem 7.2   *Which basis is basic?*
   Consider the two-level system with $\Delta = 0$ and with the basis functions $\phi_1(x)$ and $\phi_2(x)$, so that the Hamiltonian is
$$\hat{H} = \begin{pmatrix} 0 & -|V| \\ -|V| & 0 \end{pmatrix} \tag{3}$$
.
(a) Show that the eigen values are $E_\pm = \pm|V|$ and the functions are

$$\psi_-(x) = (\phi_1(x) + \phi_2(x))/\sqrt{2} \qquad \text{and} \qquad \psi_+(x) = (\phi_1(x) - \phi_2(x))/\sqrt{2}. \tag{4}$$

or equivalently
$$|\psi_- >= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad |\psi_+ >= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Now take $\psi_-(x)$ and $\psi_+(x)$ as the new set of basis functions, show that in this basis, the Hamiltonian is

$$\hat{H}' = \begin{pmatrix} -|V| & 0 \\ 0 & +|V| \end{pmatrix} \tag{5}$$

and that the eigenfunctions are now represented by

$$|\psi_- > = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_+ > = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Express $\phi_1(x)$ and $\phi_2(x)$ as column vectors in this new basis.

(d) If the initial wavefunction is $\phi_1(x)$, find the probability as a function of time for the system to remain in this wavefunction. Do your calculation in the new basis of $\psi_-(x)$ and $\psi_+(x)$. Also what is the probability as a function of time of this initial wavefunction remaining in the eigenstate $\psi_-(x)$?

Problem 7.3   *Time-dependent coupling*

Consider a two-level system driven by a sinusoidal coupling $H_{12}(t) = Ve^{i\omega t}$, where V is real. Such a time-dependent potential is used to model many driven systems; for example, electromagnetic waves which drive an electron between two atomic energy levels. The two energy levels are separated by energy $\hbar\omega_o$. Use the results given in the handout especially sections 1.5.2 and 1.5.3 to answer the following questions (do not derive, use the given formulas).

(a) If the system is driven on resonance, how long does it take the probability to go from being in the ground state to the excited state? If one increases the power of the radiation field, how does this time change.

(b) Draw figure 1.5 in the handout. Under what conditions is it difficult to drive the system from the ground state?

Problem 7.4   *Dephasing and relaxation times*

Consider the Bloch Equations with damping, Equations 18.60, 18.61, and 18.62.
(a) In the absence of any driving field ($V_{12}(t) = 0$), find the time dependence of $N(t)$, $Q(t)$, and $P(t)$. (Hint consider the quantity $Q \pm iP$.)
(b) What is the time dependence of the elements in the density matrix? Also write the density matrix in the long time limit.