6.642 Continuum Electromechanics
Fall 2008

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.642 Continuum Electromechanics

Problem Set #2	Issued: 9/10/08
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Problem 1 (Melcher, Prob. 2.16.5)

Prob. 2.16.5 A planar region, shown in Table 2.16.1, is filled by an inhomogeneous dielectric, with a permittivity that depends on x:

$$\varepsilon(x) = \varepsilon_\beta \exp 2\eta x, \quad \eta \equiv \ln(\varepsilon_\alpha / \varepsilon_\beta)/2\Delta$$

The free charge density is zero.

(a) Show that the potential distribution is

$$\tilde{\Phi} = \tilde{\Phi}^\alpha e^{-\eta(x-\Delta)} \frac{\sinh\lambda x}{\sinh\lambda\Delta} - \tilde{\Phi}^\beta e^{-\eta x} \frac{\sinh\lambda(x-\Delta)}{\sinh\lambda\Delta}$$

where

$$\lambda \equiv \sqrt{k^2 + \eta^2}$$

(b) Show that the transfer relations are

$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \\ \tilde{D}_x^\beta \end{bmatrix} = \varepsilon_\beta \lambda \begin{bmatrix} (\frac{\eta}{\lambda} - \coth\lambda\Delta) e^{\eta 2\Delta} & \frac{e^{\eta\Delta}}{\sinh\lambda\Delta} \\ \\ \frac{-e^{\eta\Delta}}{\sinh\lambda\Delta} & \frac{\eta}{\lambda} + \coth\lambda\Delta \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \\ \tilde{\Phi}^\beta \end{bmatrix}$$

Problem 2 (Melcher, Prob. 4.3.3)

Prob. 4.3.3 The moving member of an EQS device takes the form of a sheet, supporting the surface charge σ_f and moving in the z direction, as shown in Fig. P4.3.3. Electrodes on the adjacent walls constrain the potentials there.

(a) Find the force f_z on an area A of the sheet in terms of $(\hat{\Phi}^a, \hat{\sigma}_f, \hat{\Phi}^b)$.

(b) For a synchronous interaction, $\omega/k = U$. The surface charge is given by $-\sigma_o\cos[\omega t - k(z-\delta)]$ and $\Phi^a = V_o\cos(\omega t - kz)$. For even excitations $\Phi^b = \Phi^a$. Find f_z.

(c) An example of a d-c interaction is the Van de Graaff machine taken up in Sec. 4.14. With the excitations $\Phi^a = \Phi^b = -V_o\cos kz$ and $\sigma_f = \sigma_o\sin kz$, find f_z.

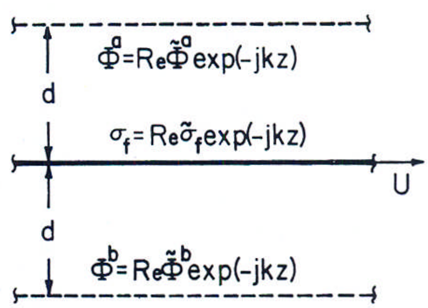
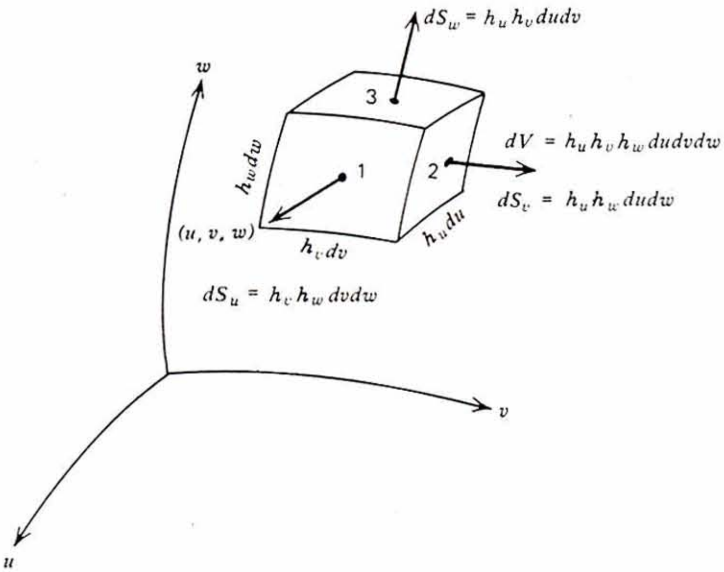


Fig. P4.3.3

Problem 3 (Zahn, Problem 23, Chapter 1)

A general right-handed orthogonal curvilinear coordinate system is described by variables (u, v, w), where

$$\mathbf{i}_u \times \mathbf{i}_v = \mathbf{i}_w$$



Since the incremental coordinate quantities du, dv, and dw do not necessarily have units of length, the differential length elements must be multiplied by coefficients that generally are a function of u, v, and w:

$$dL_u = h_u\, du, \quad dL_v = h_v\, dv, \quad dL_w = h_w\, dw$$

(a) What are the h coefficients for the Cartesian, cylindrical, and spherical coordinate systems?

(b) What is the gradient of any function $f(u, v, w)$?

(c) What is the area of each surface and the volume of a differential size volume element in the (u, v, w) space?

(d) What are the curl and divergence of the vector

$$\mathbf{A} = A_u \mathbf{i}_u + A_v \mathbf{i}_v + A_w \mathbf{i}_w ?$$

(e) What is the scalar Laplacian $\nabla^2 f = \nabla \cdot (\nabla f)$?

(f) Check your results of (b)–(e) for the three basic coordinate systems.

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6.641 FORMULA SHEET

1. DIFFERENTIAL OPERATORS IN CYLINDRICAL AND SPHERICAL COORDINATES

If r, ϕ, and z are circular cylindrical coordinates and $\hat{\imath}_r$, $\hat{\imath}_\phi$, and $\hat{\imath}_z$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\phi \frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{\imath}_z \frac{\partial U}{\partial z}$$

$$\nabla \cdot \vec{A} = div\vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = curl\vec{A} = \hat{\imath}_r \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\imath}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\imath}_z \left(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

If r, θ, and ϕ are spherical coordinates and $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\theta \frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla \cdot \vec{A} = div\vec{A} = \frac{1}{r^2}\frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{A} = curl\vec{A} = \hat{\imath}_r \left(\frac{1}{r\sin\theta}\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right) + \hat{\imath}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\right) + \hat{\imath}_\phi \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

2. SOLUTIONS OF LAPLACE'S EQUATIONS

A. Rectangular coordinates, two dimensions (independent of z):

$$\Phi = e^{kx}\left(A_1 \sin ky + A_2 \cos ky\right) + e^{-kx}\left(B_1 \sin ky + B_2 \cos ky\right)$$

(or replace e^{kx} and e^{-kx} by $\sinh kx$ and $\cosh kx$).

$$\Phi = Axy + Bx + Cy + D; (k = 0)$$

B. Cylindrical coordinates, two dimensions (independent of z):

$$\Phi = r^n\left(A_1 \sin n\phi + A_2 \cos n\phi\right) + r^{-n}\left(B_1 \sin n\phi + B_2 \cos n\phi\right)$$

$$\Phi = \ln\frac{R}{r}\left(A_1\phi + A_2\right) + B_1\phi + B_2; (n = 0)$$

C. Spherical coordinates, two dimensions (independent of ϕ):

$$\Phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta + \frac{C}{r} + D$$