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6.641 Electromagnetic Fields, Forces, and Motion, Spring 2005

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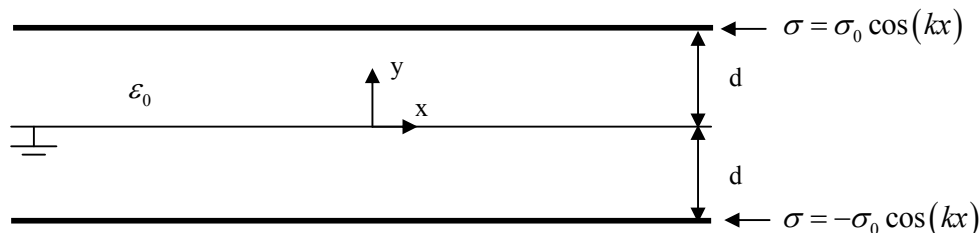
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Massachusetts Institute of Technology  
 Department of Electrical Engineering and Computer Science  
 6.641 Electromagnetic Fields, Forces, and Motion  
 Quiz 1 Solution

1. a) By placing the image surface charge as below we can find the potential in region ( $0 \leq y \leq d$ ).



$$\Phi(x, y) = \frac{\sigma_0 \cos(kx) e^{-k(d-y)}}{2\epsilon_0 k} - \frac{\sigma_0 \cos(kx) e^{-k(d+y)}}{2\epsilon_0 k} = \frac{\sigma_0 \cos(kx) e^{-kd}}{\epsilon_0 k} \sinh(ky)$$

$$\text{b) } \vec{E} = -\nabla\Phi \Rightarrow E_y = -\frac{\partial\Phi}{\partial y} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{\epsilon_0} \cosh(ky)$$

$$E_y|_{y=0} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{2\epsilon_0} (e^{ky} + e^{-ky})|_{y=0} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{\epsilon_0}$$

$$\sigma_{sf} = \epsilon_0 E_y|_{y=0} = -\sigma_0 \cos(kx) e^{-kd}$$

c) The force on the surface charge is just caused by the electrical field induced by the image charge.

$$\Phi_{image} = \frac{-\sigma_0 \cos(kx) e^{-k(d+y)}}{2\epsilon_0 k}$$

$$\vec{E} = -\nabla\Phi \Rightarrow E_{yimage} = -\frac{\partial\Phi_{image}}{\partial y} = -\frac{\sigma_0 \cos(kx) e^{-k(d+y)}}{2\epsilon_0}$$

$$\begin{aligned} f_y(x, y=d) &= \sigma_{sf}(x) E_{yimage}(x, y=d) \\ &= -\sigma_0 \cos(kx) \frac{\sigma_0 \cos(kx) e^{-2kd}}{2\epsilon_0} \\ &= -\frac{\sigma_0^2 \cos^2(kx) e^{-2kd}}{2\epsilon_0} \end{aligned}$$

2. a)  $\nabla \cdot \vec{J} = 0$ , By symmetry we just have an r component of  $\vec{J}$ .

$$\frac{1}{r} \frac{\partial}{\partial r} (rJ_r) = 0 \Rightarrow J_r = \frac{A}{r}, A \text{ is constant.}$$

$$E_r = \frac{J_r}{\sigma} = \frac{A/r}{\sigma_0 r^2 / a^2} = \frac{Aa^2}{\sigma_0 r^3}$$

$$V = \int_a^b E_r dr = \frac{Aa^2}{\sigma_0} \int_a^b \frac{1}{r^3} dr = \frac{A}{2\sigma_0} \left(1 - \frac{a^2}{b^2}\right) \Rightarrow A = \frac{2V\sigma_0}{\left(1 - \frac{a^2}{b^2}\right)}, E_r = \frac{2Va^2}{r^3 \left(1 - \frac{a^2}{b^2}\right)}$$

$$\text{b) } \nabla \cdot (\varepsilon \vec{E}) = \rho_f \Rightarrow \rho_f = \varepsilon \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = \frac{-4V\varepsilon a^2}{r^4 \left(1 - \frac{a^2}{b^2}\right)}$$

$$\text{at } r = a, \sigma_{sf}(r = a) = \varepsilon E_r |_{r=a} = \frac{2V\varepsilon}{a \left(1 - \frac{a^2}{b^2}\right)}$$

$$\text{at } r = b, \sigma_{sf}(r = b) = -\varepsilon E_r |_{r=b} = \frac{-2V\varepsilon a^2}{b^3 \left(1 - \frac{a^2}{b^2}\right)}$$

c) total volume charge

$$q_v = \int_a^b \int_0^{2\pi} \int_0^L \rho_f r dr d\phi dz = \int_a^b \int_0^{2\pi} \int_0^L \frac{-4V\varepsilon a^2}{r^3 \left(1 - \frac{a^2}{b^2}\right)} dr d\phi dz = \frac{-V\varepsilon a^2 8\pi L}{\left(1 - \frac{a^2}{b^2}\right)} \int_a^b \frac{1}{r^3} dr = -4V\varepsilon \pi L$$

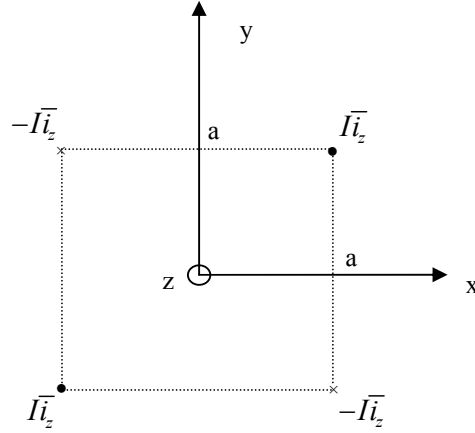
total surface charge on each electrode

$$q_s(r = a) = 2\pi a L \sigma_{sf}(r = a) = \frac{4\pi L V \varepsilon}{\left(1 - \frac{a^2}{b^2}\right)}, \quad q_s(r = b) = 2\pi b L \sigma_{sf}(r = b) = -\frac{4\pi L V \varepsilon a^2}{b^2 \left(1 - \frac{a^2}{b^2}\right)}$$

total charge in the system  $q_T = q_v + q_s(r = a) + q_s(r = b) = 0$

$$\text{d) } I = 2\pi r L J_r = \frac{4\pi V L \sigma_0}{\left(1 - \frac{a^2}{b^2}\right)} \quad \text{e) } R = \frac{V}{I} = \frac{\left(1 - \frac{a^2}{b^2}\right)}{4\pi \sigma_0 L}$$

3. a)



b) Vector potential from a single line current  $\bar{I}_z$  is:

$$A_z = \frac{-\mu_0 I}{2\pi} \ln\left(\frac{r}{r_0}\right), \text{ where } A_z(r=r_0) = 0$$

Vector potential from original current plus the three image currents shown is required so that  $H_x(x=0_+, y) = 0$  and  $H_y(x, y=0_+) = 0$  (normal component of  $\bar{H} = 0$  along the perfectly conducting surface) is:

$$A_z = \frac{-\mu_0 I}{2\pi} \left\{ \begin{aligned} & \ln\left[(x-a)^2 + (y-a)^2\right]^{1/2} - \ln\left[(x-a)^2 + (y+a)^2\right]^{1/2} \\ & + \ln\left[(x+a)^2 + (y+a)^2\right]^{1/2} - \ln\left[(x+a)^2 + (y-a)^2\right]^{1/2} \end{aligned} \right\}$$

$$= \frac{-\mu_0 I}{4\pi} \left\{ \begin{aligned} & \ln\left[(x-a)^2 + (y-a)^2\right] - \ln\left[(x-a)^2 + (y+a)^2\right] \\ & + \ln\left[(x+a)^2 + (y+a)^2\right] - \ln\left[(x+a)^2 + (y-a)^2\right] \end{aligned} \right\}$$

$$c) H_x(x, y) = \frac{1}{\mu_0} \frac{\partial A_z}{\partial y} = \frac{-I}{4\pi} \left\{ \begin{aligned} & \frac{2(y-a)}{\left[(x-a)^2 + (y-a)^2\right]} - \frac{2(y+a)}{\left[(x-a)^2 + (y+a)^2\right]} \\ & + \frac{2(y+a)}{\left[(x+a)^2 + (y+a)^2\right]} - \frac{2(y-a)}{\left[(x+a)^2 + (y-a)^2\right]} \end{aligned} \right\}$$

$$K_z(x, y=0_+) = -H_x(x, y=0_+) = \frac{I}{2\pi} \left\{ \begin{aligned} & \frac{-2a}{\left[(x-a)^2 + a^2\right]} + \frac{2a}{\left[(x+a)^2 + a^2\right]} \end{aligned} \right\}$$

$$= \frac{-Ia}{\pi} \left\{ \begin{aligned} & \frac{1}{\left[(x-a)^2 + a^2\right]} - \frac{1}{\left[(x+a)^2 + a^2\right]} \end{aligned} \right\}$$