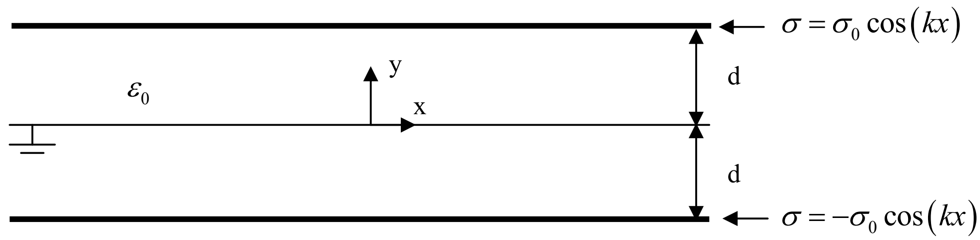
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Quiz 1 Solution

1. a) By placing the image surface charge as below we can find the potential in region ($0 \le y \le d$).



$$\Phi(x,y) = \frac{\sigma_0 \cos(kx) e^{-k(d-y)}}{2\varepsilon_0 k} - \frac{\sigma_0 \cos(kx) e^{-k(d+y)}}{2\varepsilon_0 k} = \frac{\sigma_0 \cos(kx) e^{-kd}}{\varepsilon_0 k} \sinh(ky)$$

b) $\bar{E} = -\nabla\Phi \Rightarrow E_y = -\frac{\partial \Phi}{\partial y} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{\varepsilon_0} \cosh(ky)$

$$E_y \,|_{y=0} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{2\varepsilon_0}\left(e^{ky} + e^{-ky}\right)|_{y=0} = -\frac{\sigma_0 \cos(kx) e^{-kd}}{\varepsilon_0}$$

$$\sigma_{sf} = \varepsilon_0 E_y \,|_{y=0} = -\sigma_0 \cos(kx) e^{-kd}$$

c) The force on the surface charge is just caused by the electrical field induced by the image charge.

$$\Phi_{image} = \frac{-\sigma_0 \cos(kx) e^{-k(d+y)}}{2\varepsilon_0 k}$$

$$\bar{E} = -\nabla\Phi \Rightarrow E_{yimage} = -\frac{\partial \Phi_{image}}{\partial y} = -\frac{\sigma_0 \cos(kx) e^{-k(d+y)}}{2\varepsilon_0}$$

$$\begin{aligned} f_y(x, y=d) &= \sigma_{sf}(x) E_{yimage}(x, y=d) \\ &= -\sigma_0 \cos(kx) \frac{\sigma_0 \cos(kx) e^{-2kd}}{2\varepsilon_0} \\ &= -\frac{\sigma_0{}^2 \cos^2(kx) e^{-2kd}}{2\varepsilon_0} \end{aligned}$$

2. a) $\nabla \bullet \overline{J} = 0$, By symmetry we just have an r component of $\overline{J}$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rJ_r\right) = 0 \Rightarrow J_r = \frac{A}{r}, A \text{ is constant.}$$

$$E_r = \frac{J_r}{\sigma} = \frac{A/r}{\sigma_0 r^2/a^2} = \frac{Aa^2}{\sigma_0 r^3}$$

$$V = \int_a^b E_r dr = \frac{Aa^2}{\sigma_0}\int_a^b \frac{1}{r^3}dr = \frac{A}{2\sigma_0}\left(1-\frac{a^2}{b^2}\right) \Rightarrow A = \frac{2V\sigma_0}{\left(1-\frac{a^2}{b^2}\right)},\quad E_r = \frac{2Va^2}{r^3\left(1-\frac{a^2}{b^2}\right)}$$

b) $\nabla \bullet \left(\varepsilon \overline{E}\right) = \rho_f \Rightarrow \rho_f = \varepsilon \frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right) = \frac{-4V\varepsilon a^2}{r^4\left(1-\frac{a^2}{b^2}\right)}$

at $r = a$, $\sigma_{sf}\left(r=a\right) = \varepsilon E_r \,|_{r=a} = \frac{2V\varepsilon}{a\left(1-\frac{a^2}{b^2}\right)}$

at $r = b$, $\sigma_{sf}\left(r=b\right) = -\varepsilon E_r \,|_{r=b} = \frac{-2V\varepsilon a^2}{b^3\left(1-\frac{a^2}{b^2}\right)}$

c) total volume charge

$$q_v = \int_a^b\int_0^{2\pi}\int_0^L \rho_f r dr d\phi dz = \int_a^b\int_0^{2\pi}\int_0^L \frac{-4V\varepsilon a^2}{r^3\left(1-\frac{a^2}{b^2}\right)}dr d\phi dz = \frac{-V\varepsilon a^2 8\pi L}{\left(1-\frac{a^2}{b^2}\right)}\int_a^b \frac{1}{r^3}dr = -4V\varepsilon\pi L$$

total surface charge on each electrode

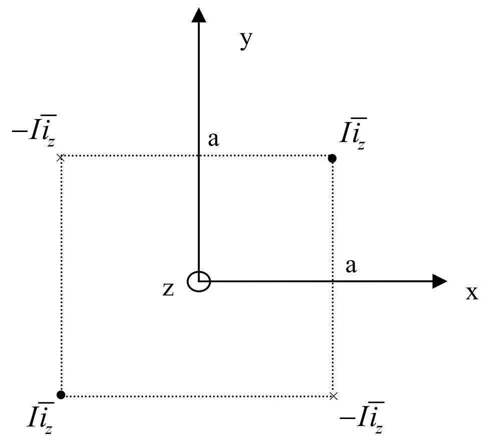
$$q_s\left(r=a\right) = 2\pi a L \sigma_{sf}\left(r=a\right) = \frac{4\pi L V\varepsilon}{\left(1-\frac{a^2}{b^2}\right)},\quad q_s\left(r=b\right) = 2\pi b L \sigma_{sf}\left(r=b\right) = -\frac{4\pi L V\varepsilon a^2}{b^2\left(1-\frac{a^2}{b^2}\right)}$$

total charge in the system $q_T = q_v + q_s\left(r=a\right) + q_s\left(r=b\right) = 0$

d) $I = 2\pi r L J_r = \frac{4\pi V L \sigma_0}{\left(1-\frac{a^2}{b^2}\right)}$

e) $R = \frac{V}{I} = \frac{\left(1-\frac{a^2}{b^2}\right)}{4\pi\sigma_0 L}$

3. a)



b) Vector potential from a single line current $I\overline{i_z}$ is:

$$A_z = \frac{-\mu_0 I}{2\pi}\ln\left(\frac{r}{r_0}\right), \text{ where } A_z(r=r_0)=0$$

Vector potential from original current plus the three image currents shown is required so that $H_x(x=0_+,y)=0$ and $H_y(x,y=0_+)=0$ (normal component of $\overline{H}=0$ along the perfectly conducting surface) is:

$$A_z = \frac{-\mu_0 I}{2\pi}\left\{\begin{array}{l}\ln\left[(x-a)^2+(y-a)^2\right]^{1/2}-\ln\left[(x-a)^2+(y+a)^2\right]^{1/2}\\ +\ln\left[(x+a)^2+(y+a)^2\right]^{1/2}-\ln\left[(x+a)^2+(y-a)^2\right]^{1/2}\end{array}\right\}$$

$$= \frac{-\mu_0 I}{4\pi}\left\{\begin{array}{l}\ln\left[(x-a)^2+(y-a)^2\right]-\ln\left[(x-a)^2+(y+a)^2\right]\\ +\ln\left[(x+a)^2+(y+a)^2\right]-\ln\left[(x+a)^2+(y-a)^2\right]\end{array}\right\}$$

c) $$H_x(x,y)=\frac{1}{\mu_0}\frac{\partial A_z}{\partial y}=\frac{-I}{4\pi}\left\{\begin{array}{l}\frac{2(y-a)}{\left[(x-a)^2+(y-a)^2\right]}-\frac{2(y+a)}{\left[(x-a)^2+(y+a)^2\right]}\\ +\frac{2(y+a)}{\left[(x+a)^2+(y+a)^2\right]}-\frac{2(y-a)}{\left[(x+a)^2+(y-a)^2\right]}\end{array}\right\}$$

$$K_z(x,y=0_+)=-H_x(x,y=0_+)=\frac{I}{2\pi}\left\{\frac{-2a}{\left[(x-a)^2+a^2\right]}+\frac{2a}{\left[(x+a)^2+a^2\right]}\right\}$$

$$=\frac{-Ia}{\pi}\left\{\frac{1}{\left[(x-a)^2+a^2\right]}-\frac{1}{\left[(x+a)^2+a^2\right]}\right\}$$