6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

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Final- Solutions 2004	
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Problem 1

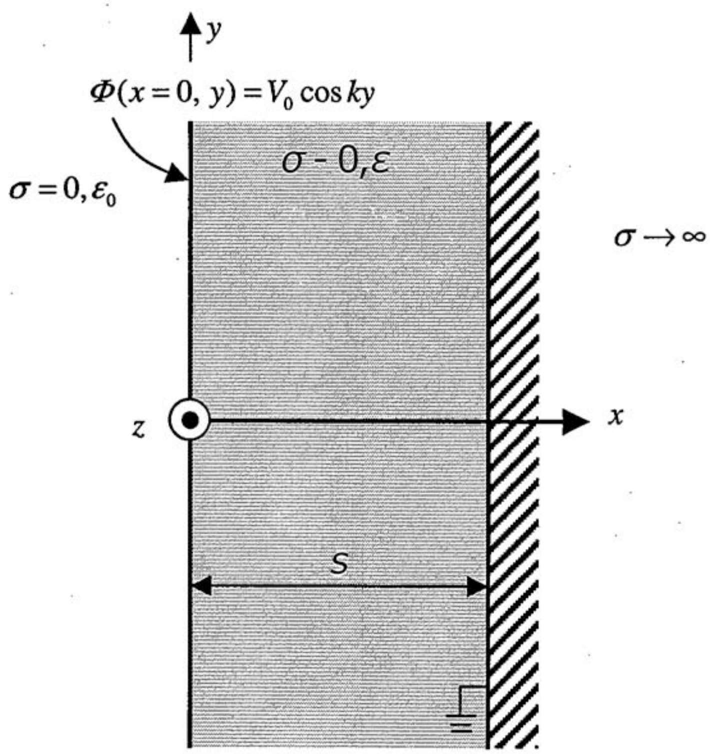

Figure 1: A potential sheet between two lossless dielectrics with a perfect conductor placed at $x = s$.

A potential sheet of infinite extent in the y and z directions is placed at $x = 0$ and has potential distribution $\Phi(x = 0, y) = V_0 \cos(ky)$. Free space with no conductivity ($\sigma = 0$) and permittivity ϵ_0 is present for $x < 0$ while for $0 < x < s$ a perfectly insulating dielectric ($\sigma = 0$) with permittivity ϵ is present. The region for $x > s$ is a grounded perfect conductor at zero potential.

A

Question: What are the potential distributions for $x < 0$ and $0 < x < s$?

Solution:

$$\Phi(x, y) = \begin{cases} V_0 \cos(ky) e^{kx} & x < 0 \\ \frac{-V_0 \sinh(k(x-s)) \cos(ky)}{\sinh(ks)} & 0 < x < s \end{cases}$$

B

Question: What are the surface charge densities at $x = 0, \sigma_f(x = 0, y)$, and at $x = s, \sigma_f(x = s, y)$?

Solution:

$$E_x = -\frac{\partial \Phi}{\partial x} = \begin{cases} -kV_0 \cos(ky) e^{kx} & x < 0 \\ \frac{kV_0 \cosh(k(x-s)) \cos(ky)}{\sinh(ks)} & 0 < x < s \end{cases}$$

$$\begin{aligned} \sigma_f(x = 0, y) &= \epsilon E_x(x = 0_+, y) - \epsilon_0 E_x(x = 0_-, y) \\ &= [\epsilon_0 + \epsilon \coth(ks)]\, kV_0 \cos(ky) \\ \sigma_f(x = s, y) &= -\epsilon E_x(x = s_-, y) \\ &= \frac{-\epsilon k V_0 \cos(ky)}{\sinh(ks)} \end{aligned}$$

C

Question: What is the force, magnitude and direction, on a section of the perfect conductor at $x = s$ that extends over the region $0 < y < \frac{\pi}{k}$ and $0 < z < D$?
Hint: $\int \cos^2 y dy = \frac{y}{2} + \frac{\sin(2y)}{4}$.

Solution:

$$\begin{aligned} \frac{F_x}{\text{area}} &= \frac{1}{2}\sigma_f E_x|_{x=s} = -\frac{1}{2}\epsilon E_x^2|_{x=s} = -\frac{1}{2}\frac{\epsilon(kV_0\cos(ky))^2}{\sinh^2(ks)} \\ F_x &= -\frac{1}{2}\frac{\epsilon k^2 V_0^2 D}{\sinh^2(ks)} \int_0^{\frac{\pi}{k}} \cos^2 ky dy = -\frac{\pi}{4}\frac{\epsilon k V_0^2 D}{\sinh^2(ks)} \end{aligned}$$

Problem 2

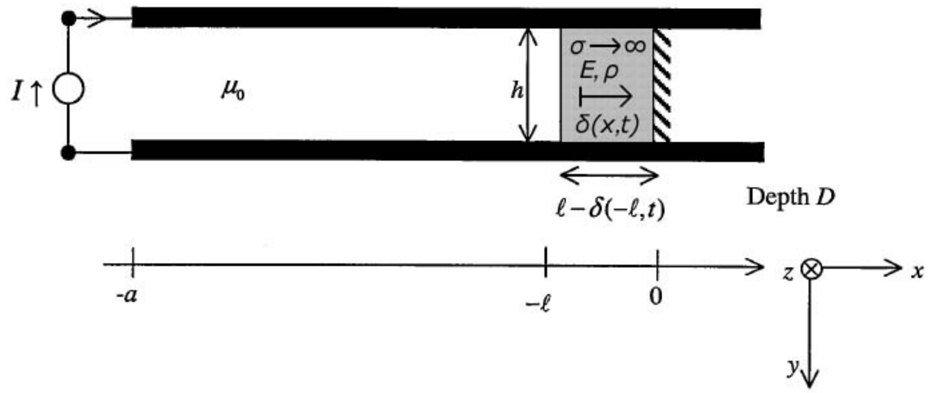

Figure 2: Parallel plate electrodes

Parallel plate electrodes with spacing h and depth D are excited by a DC current source I. An elastic rod surrounded by free space has mass density ρ, modulus of elasticity E, equilibrium length l and has infinite ohmic conductivity σ. The elastic rod end at $x = 0$ is fixed while the deflections of the rod are described as $\delta(x,t)$ and are assumed small $|\delta(x,t)| \ll l$. The rod width $l - \delta(-l,t)$ changes as I is changed because of the magnetic force. The DC current flows as a surface current on the $x = -(l - \delta(-l,t))$ end of the perfectly conducting rod.

A

Question: Calculate H_z in the free space region $-a < x < -(l - \delta(-l,t))$. Neglect fringing field effects and assume $h \ll a$ and $h \ll D$.

Solution:

$$H_z = \frac{I}{D}$$

B

Question: Using the Maxwell Stress Tensor calculate the magnetic force per unit area on the $x = -(l - \delta(-l,t))$ end of the rod.

Solution:

$$T_{xx} = \frac{1}{2}\mu_0 \left[H_x^2 - H_y^2 - H_z^2\right]\Big|_{x=-(l-\delta(-l,t))} = -\frac{\mu_0}{2}\frac{I^2}{D^2}$$

$$\frac{F_x}{\text{area}} = -T_{xx}\Big|_{x=-(l-\delta(-l,t))} = \frac{\mu_0}{2}\frac{I^2}{D^2}$$

C

Question: Calculate the steady state change in rod length $\delta(x=-l)$.

Solution:

$$\rho\frac{\partial^2\delta}{\partial t^2}\overset{0}{\nearrow} = E\frac{\partial^2\delta}{\partial x^2} \Rightarrow \delta = ax+b$$

$$\delta(x=0) = b = 0$$

$$E\left.\frac{\partial\delta}{\partial x}\right|_{x=-l} = T_{xx} = -\frac{\mu_0}{2}\frac{I^2}{D^2} = E_a$$

$$a = \frac{-\mu_0 I^2}{2D^2E} \Rightarrow \delta(x) = \frac{-\mu_0 I^2 x}{2ED^2}$$

$$\delta(-l) = \frac{\mu_0 I^2}{2ED^2}$$

D

Question: Noise creates fluctuations $\delta'(x,t)$ in longitudinal displacement. What are the natural frequencies of the rod?

Solution:

$$\rho\frac{\partial^2\delta'}{\partial t^2} = E\frac{\partial^2\delta'}{\partial x^2}, \delta'(x,t) = Re\left[\hat{\delta}(x)e^{j\omega t}\right]$$

$$-\frac{\rho\omega^2}{E}\hat{\delta}(x) = \not{E}\frac{d^2\delta}{dx^2} \Rightarrow \frac{d^2\hat{\delta}}{dx^2} + k^2\hat{\delta} = 0, k^2 = \frac{\omega^2\rho}{E}$$

$$\hat{\delta}(x) = A\sin(kx) + B\cos(kx)$$

$$\hat{\delta}(x=0) = B = 0$$

$$\left.\frac{d\hat{\delta}}{dx}\right|_{x=-l} = 0 = kA\cos(kl) \Rightarrow kl = (2n+1)\frac{\pi}{2}, n = 0,1,2$$

$$\omega_n\sqrt{\frac{\rho}{E}} = k_n = (2n+1)\frac{\pi}{2l}$$

$$\omega_n = \sqrt{\frac{E}{\rho}}\frac{\pi}{2l}(2n+1), n = 0,1,2$$

Problem 3

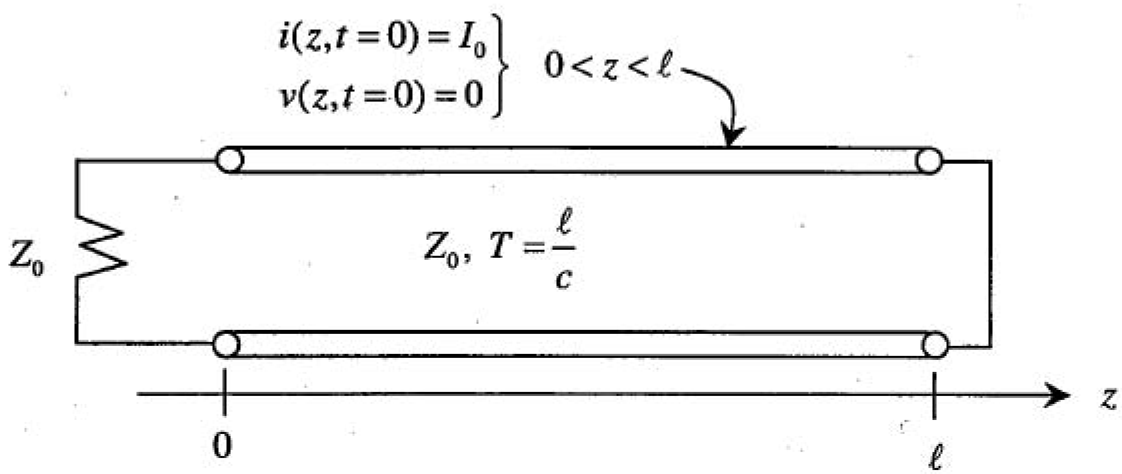

Figure 3: An electrical transmission line

An electrical transmission line of length l has characteristic impedance Z_0. Electromagnetic waves can travel on the line at speed c, so that the time to travel one-way over the line length l is $T = \frac{l}{c}$. The line is matched at $z = 0$ and is short circuited at $z = l$. At time $t = 0$, a lightning bolt strikes the entire line so that there is a uniform current along the line but with zero voltage:

$$i(z, t = 0) = I_0 \qquad 0 < z < l$$
$$v(z, t = 0) = 0 \qquad 0 < z < l$$

Since the voltage and current obey the telegrapher's relations:

$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}, c = \frac{1}{\sqrt{LC}}$$
$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}, Z_0 = \sqrt{\frac{L}{C}}$$

the voltage and current along the line are related as

$$v + iZ_0 = c_+ \text{ on } \frac{dz}{dt} = c \Rightarrow v = \frac{c_+ + c_-}{2}$$
$$v - iZ_0 = c_- \text{ on } \frac{dz}{dt} = -c \Rightarrow iZ_0 = \frac{c_+ - c_-}{2}$$

A

Question: The solutions for $v(z, t)$ and $i(z, t)$ can be found using the method of characteristics within each region shown below (see exam questions). Within regions 1-9 give the values of c_+, c_-, v and iZ_0.

Solution: See Figure 4.

B

Question: Plot $v(z, t = \frac{T}{4})$ and $i(z, t = \frac{T}{4})$.

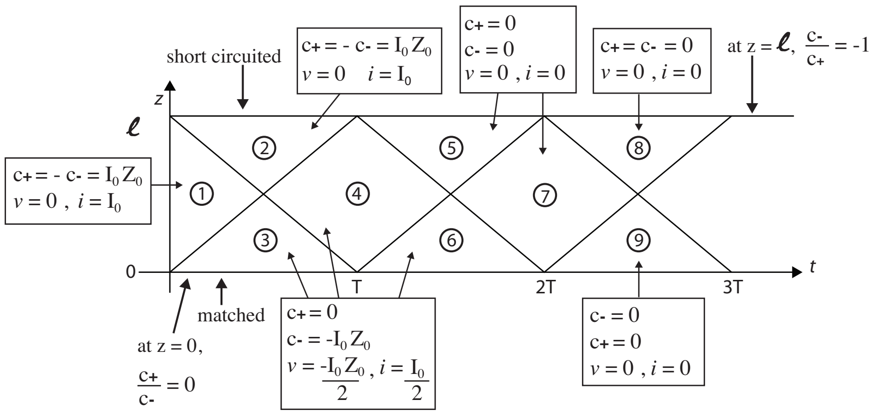

Figure 4: Values of c_+, c_-, v and iZ_0 in regions 1-9. (Image by MIT OpenCourseWare.)

Solution:

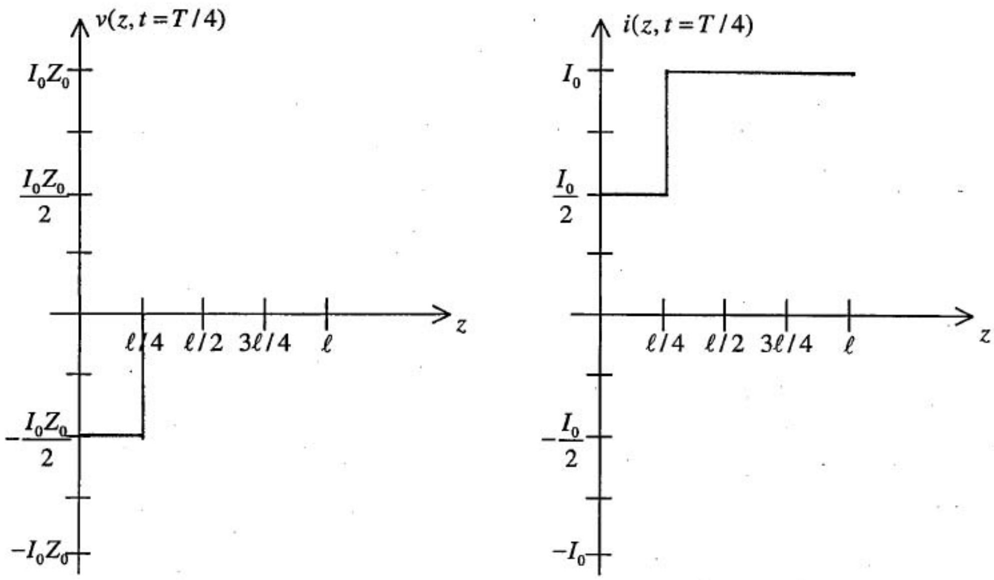

Figure 5: plots of $v(z, t = \frac{T}{4})$ and $i(z, t = \frac{T}{4})$ for problem 2, part B (Image by MIT OpenCourseWare.)

C

Question: How long a time does it take for the transmission line to have $v(z,t) = 0$ and $i(z,t) = 0$ everywhere for $0 < z < l$ for all further time?

Solution: 2T

Problem 4

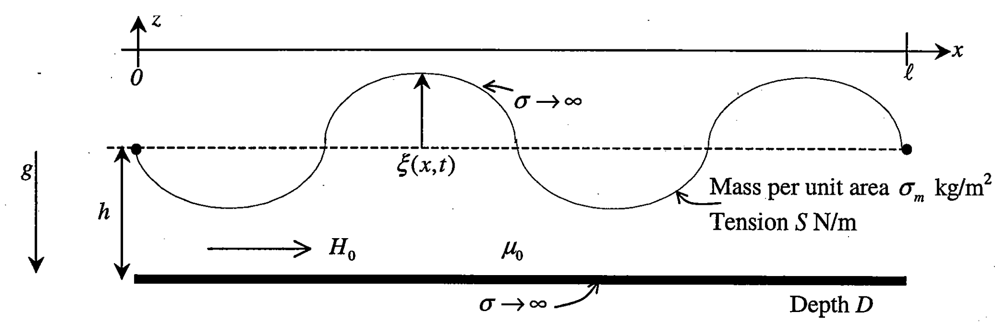

Figure 6: A perfectly conducting membrane stressed from below by magnetic field $H_0\bar{i_x}$

A perfectly conducting membrane of depth D with mass per unit area σ_m and tension S is a distance h above a rigid perfect conductor. The membrane and rigid conductor are in free space and support currents such that when the membrane is flat, $\xi(x,t) = 0$, the static uniform magnetic field intensity is H_0. As the membrane deforms, the flux through the region between membrane and rigid conductor is conserved. The system is in a downward gravity field with gravitational acceleration $\bar{g} = -g\bar{i_z}$. The membrane deflection has no dependence on y and is fixed at its two ends at $x = 0$ and $x = l$.

A

Question: Assuming that $\xi(x,t) \ll h$ and that the only significant magnetic field component is x directed, how is $H_x(x,t)$ approximately related to $\xi(x,t)$ to linear terms in $\xi(x,t)$?

Solution:

$$H_x(h+\xi) = H_0 h \Rightarrow H_x = \frac{H_0 h}{h+\xi} = \frac{H_0}{1+\frac{\xi}{h}} \approx H_0(1 - \frac{\xi}{h})$$

B

Question: Using the Maxwell Stress tensor and the result of part(a), to linear terms in small displacement $\xi(x,t)$, what is the z directed magnetic force per unit area, F_z, on the membrane?

Solution:

$$T_{zz} = \frac{\mu_0}{2}\left(\cancelto{0}{H_z^2} - H_x^2 - \cancelto{0}{H_y^2}\right) = -\frac{\mu_0}{2}H_0^2\left(1-\frac{\xi}{h}\right)^2 \approx -\frac{\mu_0 H_0^2}{2}\left(1-\frac{2\xi}{h}\right)$$

$$F_z = -T_{zz} = \frac{\mu_0 H_0^2}{2}\left(1-\frac{2\xi}{h}\right)$$

C

<u>Question:</u> To linear terms in small displacement $\xi(x,t)$, express the membrane equation of motion in the form

$$a\frac{\partial^2\xi}{\partial t^2} = b\frac{\partial^2\xi}{\partial x^2} + c\xi + d$$

What are $a, b, c,$ and d?

<u>Solution:</u>

$$\begin{aligned}\sigma_m\frac{\partial^2\xi}{\partial t^2} &= S\frac{\partial^2\xi}{\partial x^2} + F_z - \sigma_m g \\ &= S\frac{\partial^2\xi}{\partial x^2} + \mu_0\frac{H_0^2}{2}\left(1 - \frac{2\xi}{h}\right) - \sigma_m g\end{aligned}$$

$$a = \sigma_m, b = S, c = -\frac{\mu_0 H_0^2}{h}, d = \frac{\mu_0 H_0^2}{2} - \sigma_m g$$

D

<u>Question:</u> What value of H_0 is needed so that in static equilibrium the membrane has no sag, $\xi(x,t) = 0$.

<u>Solution:</u>

$$\frac{\mu_0 H_0^2}{2} = \sigma_m g \Rightarrow H_0 = \left[\frac{2\sigma_m g}{\mu_0}\right]^{\frac{1}{2}}$$

E

<u>Question:</u> About the equilibrium of part (d), what is the $\omega{-}k$ dispersion relation for membrane deflections of the form

$$\xi(x,t) = Re\left[\hat{\xi}e^{j(\omega t - kx)}\right]?$$

Solve for k as a function of ω and system parameters.

<u>Solution:</u>

$$\begin{aligned}-\sigma_m\omega^2 &= -Sk^2 - \frac{\mu_0 H_0^2}{h} \\ k^2 &= \frac{\sigma_m}{S}\omega^2 - \frac{\mu_0 H_0^2}{hS} \\ k &= \pm\left[\frac{\sigma_m\omega^2}{S} - \frac{\mu_0 H_0^2}{hS}\right]^{\frac{1}{2}} = \pm k_0 \\ k_0 &= +\left[\frac{\sigma_m\omega^2}{S} - \frac{\mu_0 H_0^2}{hS}\right]^{\frac{1}{2}}\end{aligned}$$

F

Question: Using all the values of k found in part (e), find a superposition of solutions of the form of $\xi(x,t)$ given in (e) that satisfy the zero deflection boundary conditions at the ends of the membrane at $x=0$ and $x=l$. What are the allowed values of k?

Solution:

$$\begin{aligned}
\xi(x,t) &= Re\left[e^{j\omega t}\left[\hat{\xi}_1 e^{-jk_0x} + \hat{\xi}_2 e^{+jk_0x}\right]\right] \\
\xi(x=0,t) = 0 &= Re\left[e^{j\omega t}\left[\hat{\xi}_1 + \hat{\xi}_2\right]\right] \Rightarrow \hat{\xi}_2 = -\hat{\xi}_1 \\
\xi(x,t) &= Re\left[e^{j\omega t}\hat{\xi}_1\left[e^{-jk_0x} - e^{jk_0x}\right]\right] \\
&= Re\left[e^{j\omega t}\hat{\xi}_1(-2j)\sin(k_0x)\right] \\
\xi(x=l,t) = 0 &= Re\left[e^{j\omega t}\hat{\xi}_1(-2j)\sin(k_0l)\right] = 0 \\
\sin(k_0l) = 0 &\Rightarrow k_0l = n\pi, n = 1, 2, \ldots
\end{aligned}$$

G

Question: Is this system always stable or under what conditions can it be unstable? When stable, what are the natural frequencies and if unstable what are the growth rates of the instability?

Solution:

$$\begin{aligned}
\omega^2 &= \frac{S}{\sigma_m}k^2 + \frac{\mu_0 H_0^2}{h\sigma_m} \\
\omega_n &= \left[\frac{S}{\sigma_m}\left(\frac{n\pi}{l}\right)^2 + \frac{\mu_0 H_0^2}{h\sigma_m}\right]^{\frac{1}{2}} \quad \text{Always stable as } \omega_n \text{ real.}
\end{aligned}$$