# Green's functions for planarly layered media (continued)

Massachusetts Institute of Technology
6.635 lecture notes

We shall here continue the treatment of multilayered media Green's functions, starting from the TE/TM decomposition we have presented in the previous document.

For the sake of illustration, let us consider a one layer medium with a reflection and transmission region, as shown in Fig. 1.
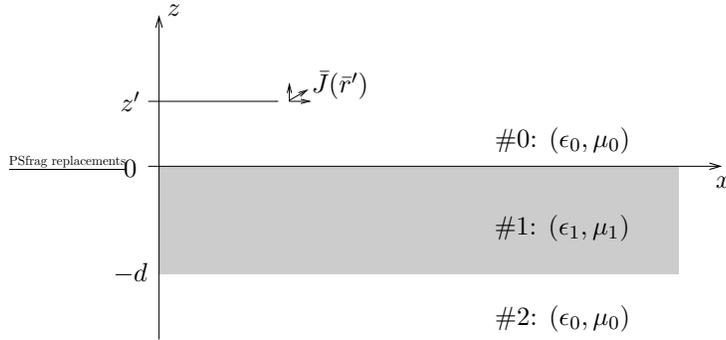
Figure 1: Geometry of the problem.

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We have shown before that the Green's functions in various layers are expressed as:

$$\bar{\bar{G}}_{\ell 0} = \frac{i}{8\pi^2} \iint d\bar{k}_\perp \frac{1}{k_{0_z}} \left[\bar{K}_e(k_{\ell_z})\hat{e}(-k_{0_z}) e^{-i\bar{K}\cdot\bar{r}'} + \bar{K}_h(k_{\ell_z})\hat{h}(-k_{0_z}) e^{-i\bar{K}\cdot\bar{r}'} \right], \tag{1}$$

where

$$\ell = 0,\; z < z' : \qquad \bar{K}_e(k_{0_z}) = \hat{e}(-k_{0_z}) e^{i\bar{K}\cdot\bar{r}} + R^{TE}\hat{e}(k_{0_z}) e^{i\bar{k}\cdot\bar{r}}, \tag{2a}$$

$$\bar{K}_h(k_{0_z}) = \hat{h}(-k_{0_z}) e^{i\bar{K}\cdot\bar{r}} + R^{TM}\hat{h}(k_{0_z}) e^{i\bar{k}\cdot\bar{r}}, \tag{2b}$$

$$\ell = 1 : \qquad \bar{K}_e(k_{1_z}) = A\,\hat{e}(k_{1_z}) e^{i\bar{k}_1\cdot\bar{r}} + B\,\hat{e}(-k_{1_z}) e^{i\bar{K}_1\cdot\bar{r}}, \tag{2c}$$

$$\bar{K}_h(k_{1_z}) = C\,\hat{h}(k_{1_z}) e^{i\bar{k}_1\cdot\bar{r}} + D\,\hat{h}(-k_{1_z}) e^{i\bar{K}_1\cdot\bar{r}}, \tag{2d}$$

$$\ell = 2 : \qquad \bar{K}_e(k_{2_z}) = T^{TE}\,\hat{e}(-k_{0_z}) e^{i\bar{K}\cdot\bar{r}}, \tag{2e}$$

$$\bar{K}_h(k_{2_z}) = T^{TM}\,\hat{h}(-k_{0_z}) e^{i\bar{K}\cdot\bar{r}}. \tag{2f}$$

By satisfying the boundary conditions at $z = 0$ and $z = -d$, we obtain the following system (for TE waves, TM waves can be solved similarly):

$$1 + R^{TE} = A + B \tag{3a}$$

$$\frac{k_{0_z}}{\mu_0} \left(-1 + R^{TE}\right) = \frac{k_{1_z}}{\mu_1} \left(A - B\right), \tag{3b}$$

$$A e^{-ik_{1_z}d} + B e^{ik_{1_z}d} = T^{TE} e^{ik_{0_z}d}, \tag{3c}$$

$$\frac{k_{1_z}}{\mu_1} \left(A e^{-ik_{1_z}d} - B e^{ik_{1_z}d}\right) = -\frac{k_{0_z}}{\mu_0} T^{TE} e^{ik_{0_z}d}. \tag{3d}$$

Upon solving, we obtain:

$$R^{TE} = \frac{1 - e^{2ik_{1_z}d}}{1 + R_{01}^{TE} R_{10}^{TE} e^{2ik_{1_z}d}} R_{01}^{TE}, \tag{4a}$$

$$T^{TE} = \frac{4 e^{i(k_{1_z} - k_{0_z})d}}{\left(1 + p_{01}^{TE}\right)\left(1 + p_{10}^{TE}\right)\left(1 + R_{01}^{TE} R_{10}^{TE} e^{2ik_{1_z}d}\right)}. \tag{4b}$$

$$R^{TM} = \frac{1 - e^{2ik_{1_z}d}}{1 + R_{01}^{TM} R_{10}^{TM} e^{2ik_{1_z}d}} R_{01}^{TM}, \tag{5a}$$

$$T^{TM} = \frac{4 e^{i(k_{1_z} - k_{0_z})d}}{\left(1 + p_{01}^{TM}\right)\left(1 + p_{10}^{TM}\right)\left(1 + R_{01}^{TM} R_{10}^{TM} e^{2ik_{1_z}d}\right)}. \tag{5b}$$

$$A = \frac{1 - p_{10}^{TE}}{2} \frac{1 + R_{01}^{TE} R_{10}^{TE}}{1 + R_{01}^{TE} R_{10}^{TE} e^{2ik_{1_z}d}} e^{2ik_{1_z}d}, \tag{6a}$$

$$B = \frac{1 + R_{01}^{TE}}{1 + R_{01}^{TE} R_{10}^{TE} e^{2ik_{1_z}d}}, \tag{6b}$$

$$C = \frac{\mu_1 k_0}{\mu_0 k_1} \frac{2 R_{10}^{TM}}{\left(1 + p_{01}^{TM}\right)\left(1 + R_{01}^{TM} R_{10}^{TM} e^{2ik_{1_z}d}\right)} e^{2ik_{1_z}d}, \tag{6c}$$

$$D = \frac{\mu_1 k_0}{\mu_0 k_1} \frac{2}{\left(1 + p_{01}^{TM}\right)\left(1 + R_{01}^{TM} R_{10}^{TM} e^{2ik_{1_z}d}\right)}. \tag{6d}$$

1 Transmission line analogy for multilayered media

Let us consider a plane wave incident from region 0, with its plane of incidence parallel to the (xz) plane. The medium it is incident upon is multilayered.

All fields vectors are independent on y, so that $\frac{\partial}{\partial y} = 0$ in Maxwell's equations. Thus, we can decompose the electromagnetic field into TE/TM components.

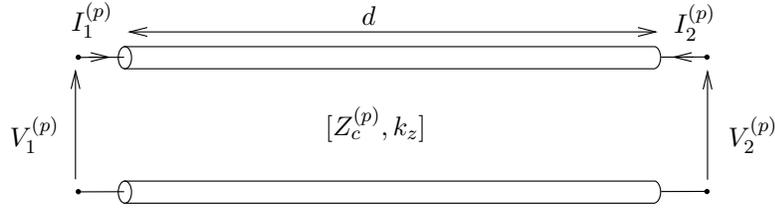We get, in region ℓ (for TE waves):

$$E_{\ell y} =[A_\ell e^{ik_z z} + B_\ell e^{-ik_z z}]\, e^{ik_x x}\,, \tag{7a}$$

$$H_{\ell x} = -\,\frac{k_{z\ell}}{\omega\mu_\ell}\,[A_\ell e^{ik_z z} - B_\ell e^{-ik_z z}]\, e^{ik_x x}\,, \tag{7b}$$

$$H_{\ell z} =\frac{k_x}{\omega\mu_\ell}\,[A_\ell e^{ik_z z} - B_\ell e^{-ik_z z}]\, e^{ik_x x}\,. \tag{7c}$$

FOLLOWING STANDARD NOTATION IN TRANSMISSION LINE THE-
ORY, WE SHALL USE HERE THE j NOTATION!!

From 6.630, we know that a transmission line is characterized by its length d, characteristic
impedance $Z_c^{(p)}$, and wavenumber k_z, as defined in Fig. 2.

Figure 2: Transmission line circuit.

In a source-free region, the transmission line equations are written as

$$\frac{\partial}{\partial z}V^{(p)} = -\,jk_z Z_c^{(p)}I^{(p)}\,, \tag{8a}$$

$$\frac{\partial}{\partial z}I^{(p)} = -\,jk_z\frac{1}{Z_c^{(p)}}V^{(p)}\,, \tag{8b}$$

where p refers to the polarization (TE or TM), and

$$Z_c^{TE} = \frac{\omega\mu}{k_z}\,, \tag{9a}$$

$$Z_c^{TM} = \frac{k_z}{\omega\epsilon}\,. \tag{9b}$$

The solution to Eqs. (8) is:

$$V^{(p)} =A^{(p)}e^{-ik_z z} + B^{(p)}e^{ik_z z}\,, \tag{10a}$$

$$I^{(p)} =\frac{1}{Z_c^{(p)}}[A^{(p)}e^{-ik_z z} - B^{(p)}e^{ik_z z}]\,, \tag{10b}$$

As it can be seen, there is a direct analogy between the voltage/current in a transmission
line and the components of the electric and magnetic fields. For example, referring back to
Eq. (7), we write:

$$E_{\ell y} = V_\ell^{TE}(z)e^{ik_x x}, \tag{11a}$$

$$H_{\ell x} = -I_\ell^{TE}(z)e^{ik_x x}. \tag{11b}$$

Therefore, the problem of computing the fields in multilayered media in the spectral domain comes down to determining the voltage/current in equivalent transmission line network. The analogy is illustrated in Fig. 3.
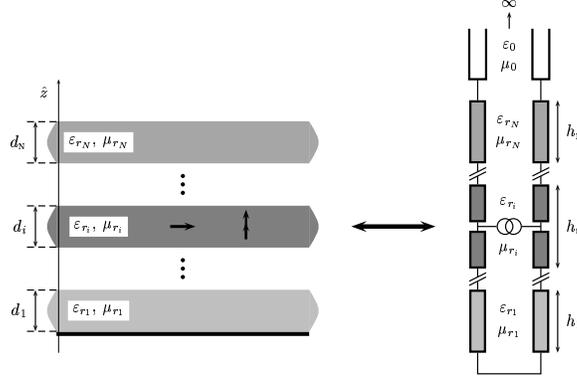
Figure 3: Transmission line analogy for horizontal electric source and vertical magnetic source. Other cases are obtained by duality.

The treatment of the source will not be demonstrated here and we shall just state the final results (details can be found in the literature). Thus, depending on the source type and orientation, different generators will have to be placed in the transmission line network. The various cases are (magnetic sources can be obtained by duality):

Horizontal electric source	current generator	value: $1/2\pi$
Vertical electric source	voltage generator	value: $1/2\pi$

The algorithm is therefore as follows:

1. Write the field components in terms of voltage and current. Locate the source and observation point.

2. Compute the equivalent transmission line network. Locate the source (type and position) and observation.

3. Starting from the source replace all the layers above and below the source by equivalent impedances (see Fig. 4).

   To do this, start with the extreme boundary conditions and propagate back to the source using:
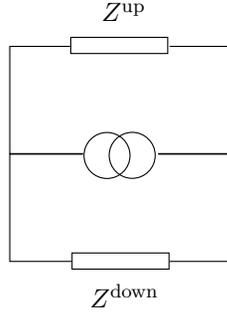
Figure 4: Equivalent upper and lower impedance.

$$Z_{in}(p) = Z_c^{(p)} \frac{Z_L^{(p)} + jZ_c^{(p)} \tan(k_z d)}{Z_c^{(p)} + jZ_L^{(p)} \tan(k_z d)} . \tag{12}$$

4. Using standard circuit theory, compute $V^{(p)}$ and $I^{(p)}$ at the upper and lower limits.

5. Propagate $V^{(p)}$ and $I^{(p)}$ until the observation point using

$$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} \cos(k_z d) & -jZ_c^{(p)} \sin(k_z d) \\ -j/Z_c^{(p)} \sin(k_z d) & \cos(k_z d) \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} . \tag{13}$$

6. Get the fields in the spectral domain.

2 Coming back to the space domain: Sommerfeld integral

IN THE REST OF THESE NOTES WE COME BACK TO THE i NOTATION!!

To come back to the space domain, we need to evaluate the inverse Fourier transform. A typical integral we have to perform is:

$$f(\bar{r}) = \frac{1}{(2\pi)^2} \iint dk_x dk_y \, \tilde{f}(k_x, k_y) \, e^{ik_x x} \, e^{ik_y y} . \tag{14}$$

By symmetry of the problem (x and y axis are equivalent), we can make a change of variables and integrate one integral analytically. The proper change of variables is the following:

$$k_x = k_\rho \cos k_\phi , \qquad\qquad x = \rho \cos \phi , \tag{15}$$

$$k_y = k_\rho \sin k_\phi , \qquad\qquad y = \rho \sin \phi . \tag{16}$$

We can transform the exponential part as:

$$e^{ik_x x} \, e^{ik_y y} = e^{i[k_\rho \rho \cos k_\phi \cos \phi + k_\rho \rho \sin k_\phi \sin \phi]} = e^{ik_\rho \rho \cos(k_\phi - \phi)} , \tag{17}$$

so that

$$f(\bar{r}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} dk_\phi \int_0^\infty dk_\rho \, k_\rho \, \tilde{f}(k_\rho, k_\phi) \, e^{ik_\rho \rho \cos(k_\phi - \phi)} . \tag{18}$$

By rotational symmetry, $\tilde{f}(k_\rho, k_\phi) = \tilde{f}(k_\rho)$. In addition, we can expand the exponential part using the following identity:

$$e^{ik_\rho\rho\cos(k_\phi-\phi)} = e^{i\beta\cos\theta} = \sum_{m=-\infty}^{\infty} (-i)^m \, J_m(\beta) \, e^{im\theta}\,, \tag{19}$$

(where β and θ have just been defined to simplify the notation in the identity and have no connection to physical parameters).

As we can see, the exponential function is the only term depending on k_ϕ. Performing the integration, we get:

$$\int_{-\phi}^{2\pi-\phi} e^{im\theta} = \begin{cases} 0 & \text{if } m \neq 0\,, \\ 2\pi & \text{if } m = 0\,. \end{cases} \tag{20}$$

Therefore, we end up with

$$f(\bar{r}) = \frac{1}{2\pi} \int_0^\infty \mathrm{d}k_\rho \, k_\rho \, \tilde{f}(k_\rho) \, J_0(k_\rho\rho)\,, \tag{21}$$

which is known as a Sommerfeld integral.

Note however that in Eq. (7b), the kernel is function of k_x also, which adds a $k_\rho \cos k_\phi$ term in the integral, so that the integral in k_ϕ cannot be performed exactly as shown above. However, the generalization to this case is straightforward. Upon performing the same expansion of the Bessel function, we see that this time the non-vanishing contribution will come from the $m = 1$ term (or ± 1 depending on how the integral is written).

Without further details, we generalize the definition of Sommerfeld integral to the nth order as:

$$S_n[\tilde{f}] = \int_0^\infty \mathrm{d}k_\rho J_n(k_\rho\rho)k_\rho^{n+1}\tilde{f}(k_\rho)\,, \tag{22}$$

and the transformation from spectral to spatial can be summarized as follows (where $\tilde{A}$ is a function of k_ρ only):

SPECTRAL DOMAIN	SPACE DOMAIN
$\tilde{G} = \tilde{A}$	$G = S_0[\tilde{A}]$
$\tilde{G} = -ik_x\tilde{A}$	$G = -\cos\phi \, S_1[\tilde{A}]$
$\tilde{G} = -ik_y\tilde{A}$	$G = -\sin\phi \, S_1[\tilde{A}]$
$\tilde{G} = -k_x^2\tilde{A}$	$G = \dfrac{\cos 2\phi}{\rho} S_1[\tilde{A}] - \cos^2\phi \, S_0[k_\rho^2\tilde{A}]$
$\tilde{G} = -k_y^2\tilde{A}$	$G = -\dfrac{\cos 2\phi}{\rho} S_1[\tilde{A}] - \sin^2\phi \, S_0[k_\rho^2\tilde{A}]$
$\tilde{G} = -k_xk_y\tilde{A}$	$G = \dfrac{\sin 2\phi}{\rho} S_1[\tilde{A}] - \dfrac{1}{2}\sin 2\phi \, S_0[k_\rho^2\tilde{A}]$

3 Numerical evaluation of Sommerfeld integrals

Sommerfeld integrals are difficult (but not impossible) to evaluate for two reasons:

1. The spectral kernel can present poles (and in general does).

2. They have an oscillatory tail.

Fortunately these two problems appear in essentially two distinct regions.

3.1 Poles of the Green's function

It can be shown (out of the scope here) that the poles of the Green's functions are associated with propagating waves (*e.g.* in single-mode regime, the Green's function has only one pole). Therefore, we must have at least one value of ℓ (the index of the region) where k_{ℓ_z} is real, *i.e.*

$$k_{\ell_z} = \sqrt{k_l^2 - k_\rho^2} = \sqrt{k_0^2 \epsilon_\ell \mu_\ell - k_\rho^2} \tag{23}$$

must be real. Since this needs to happen in at least one layer, it yields the condition:

$$k_\rho < k_0 \max_l(\sqrt{\epsilon_\ell \mu_\ell}), \tag{24}$$

which puts an upper limit for the location of the poles. Although it can also be shown that poles have to correspond to $k_\rho > k_0$, we do not need this constraint here and we can limit ourselves to the interval $[0, k_0 \max_l(\sqrt{\epsilon_\ell \mu_\ell})]$.

In the lossless situation, the poles lie on the real axis, which renders the integral impossible to evaluate as is. We therefore need to deform the contour in the complex k_ρ-plane:

- At infinity, convergence is ensured by Sommerfeld's radiation condition.

- On $[0, k_0 \max_l(\sqrt{\epsilon_\ell \mu_\ell})]$, we perform the integration over an ellipse, as shown in Fig. 5.
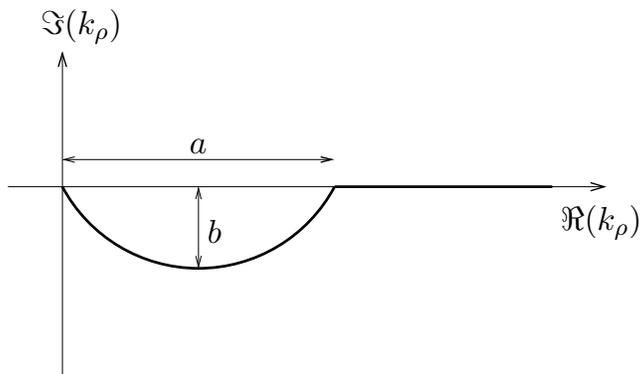
Figure 5: Contour deformation in the complex k_ρ-plane.

3.2   Oscillatory tail

The problem in this case is not the divergence of the integral (it does not diverge) bu the convergence, which is very slow because of the oscillatory behavior of the kernel. Yet, we can apply acceleration techniques to sum the series (these acceleration techniques belong to the family of *extrapolation* techniques).

For the sake of illustration, we can mention Euler's transformation, which is one of the best known acceleration technique:

$$S'_n = \frac{S_n + S_{n+1}}{2} \,, \tag{25}$$

where $S_n = \sum_{i=0}^{\infty} u_i$ is the partial sum (u_i being the terms of the original series).

By applying the formula repeatedly, we get:

$$S_n^{k+1} = \frac{S_n^{(k)} + S_{n+1}^{(k)}}{2} \,. \tag{26}$$

A direct improvement of Euler technique is to weight the partial sums and write:

$$S'_n = \frac{w_n S_n + w_{n+1} S_{n+1}}{w_n + w_{n+1}} \,. \tag{27}$$

In our specific situation, we need to evaluate integrals of the form:

$$S = \int_{\alpha}^{\infty} f(x) \mathrm{d}x \,, \tag{28}$$

where α is related to $k_0 \max_l(\sqrt{\epsilon_\ell \mu_\ell})$. In order to transform this integral in a series, we apply the approach of "integration then summation". Thus, we define:

$$u_i = \int_{x_{i-1}}^{x_i} f(x) \mathrm{d}x \,, \tag{29a}$$

$$S = \sum_{i=0}^{\infty} u_i \,, \tag{29b}$$

$$S_n = \sum_{i=0}^{n} u_i \,. \tag{29c}$$

The break points x_i have to be well chosen, and may for example be chosen based on the asymptotic behavior of f. If we refer back to Sommerfeld integrals, we can take the asymptotic expansion of the Bessel function:

$$J_\nu(k_\rho \rho) \simeq \sqrt{\frac{2}{\pi k_\rho \rho}} \, \cos(k_\rho \rho - \nu \frac{\pi}{2} - \frac{\pi}{4}) \,. \tag{30}$$

Hence, the easiest/simplest choice of break points will be

$$x_n = k_{\rho_n} = x_0 + n\frac{\pi}{\rho} \,, \tag{31}$$

where x_0 is the first break point greater than α. We will then approximate S by S_n as

$$S_n = \int_\alpha^{x_n} f(x)\mathrm{d}x\,. \tag{32}$$

The problem of Sommerfeld's integrals is that the remainder

$$r_n = S - S_n = -\int_{x_n}^\infty f(x)\mathrm{d}x \tag{33}$$

decays slowly, so that we want to accelerate the series from which S_n is evaluated (see Eq. (29c)). For Sommerfeld-type integrals, we write the generic form as:

$$I = \int_\alpha^\infty g(k_\rho\rho)\,f(k_\rho)\,\mathrm{d}k_\rho\,, \tag{34}$$

and the partial integral as

$$I(n) = \int_\alpha^{x_n} g(k_\rho\rho)\,f(k_\rho)\,\mathrm{d}k_\rho\,. \tag{35}$$

The remainder is therefore

$$I - I(n) = \int_{x_n}^\infty g(k_\rho\rho)\,f(k_\rho)\,\mathrm{d}k_\rho\,. \tag{36}$$

This integral can be expanded into an infinite series of inverse powers of ρ by integration by part. For example, with $g(k_\rho\rho) = e^{ik_\rho\rho}$ (which is a generic form for Sommerfeld integrals), we can write:

$$I - I_0(n) = \frac{i}{\rho}\,e^{ia_n\rho}\left[f_n + \frac{i}{\rho}f_n' + (\frac{i}{\rho})^2 f_n'' + \dots \right], \tag{37}$$

where $f_n = f(x_n)$, $f_n' = f'(x_n)$, etc. Note that if the right-hand side term converges (and it does in our case), the dominant term is $\mathcal{O}(\rho^{-1})$. Yet, if we now construct

$$I_1(n) = \frac{f_{n+1}I_0(n) + f_n I_0(n+1)}{f_n + f_{n+1}}\,, \tag{38}$$

it appears to be a better estimate of I since the error is in $\mathcal{O}(\rho^{-2})$.

If, in addition,

$$f(k_\rho) \sim C k_\rho^\alpha\, e^{-\beta k_\rho}\,, \tag{39}$$

we can approximate f_n and f_{n+1} and write

$$I_1(n) = \frac{I_0(n) + \eta_0 I_0(n+1)}{1 + \eta_0}\,, \tag{40}$$

where $\eta_0 = [n/(n+1)]^\alpha\, e^{\beta\pi/\rho}$. At higher orders, a better approximation is given by:

$$I_2(n) = \frac{I_1(n) + \eta_1 I_1(n+1)}{1 + \eta_1}\,, \tag{41}$$

where $\eta_1 = [(n-1/2)/(n+1/2)]^{\alpha-2}\, e^{\beta\pi/\rho}$.

In practical applications, the parameters α and β may have to be adjusted for an optimum convergence.

The technique presented briefly here is known as the "*weighted average*" method, and more details can be found in the literature under this keyword.