Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 6 Solutions
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Problem 6.1

Here we begin the analysis of quantum linear transformations by treating the single-frequency quantum theory of the beam splitter.

(a) It is straightforward to verify the energy conservation property of the beam splitter's input-output relation. We have that,

$$\begin{aligned}\hat{a}^\dagger_{\text{OUT}}\hat{a}_{\text{OUT}} &= (\sqrt{\epsilon}\,\hat{a}^\dagger_{\text{IN}} + \sqrt{1-\epsilon}\,\hat{b}^\dagger_{\text{IN}})(\sqrt{\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{1-\epsilon}\,\hat{b}_{\text{IN}}) \\ &= \epsilon\hat{a}^\dagger_{\text{IN}}\hat{a}_{\text{IN}} + (1-\epsilon)\hat{b}^\dagger_{\text{IN}}\hat{b}_{\text{IN}} + \sqrt{\epsilon(1-\epsilon)}(\hat{a}^\dagger_{\text{IN}}\hat{b}_{\text{IN}} + \hat{b}^\dagger_{\text{IN}}\hat{a}_{\text{IN}}).\end{aligned}$$

Similarly, we have that,

$$\begin{aligned}\hat{b}^\dagger_{\text{OUT}}\hat{b}_{\text{OUT}} &= (-\sqrt{1-\epsilon}\,\hat{a}^\dagger_{\text{IN}} + \sqrt{\epsilon}\,\hat{b}^\dagger_{\text{IN}})(-\sqrt{1-\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{\epsilon}\,\hat{b}_{\text{IN}}) \\ &= (1-\epsilon)\hat{a}^\dagger_{\text{IN}}\hat{a}_{\text{IN}} + \epsilon\hat{b}^\dagger_{\text{IN}}\hat{b}_{\text{IN}} - \sqrt{\epsilon(1-\epsilon)}(\hat{a}^\dagger_{\text{IN}}\hat{b}_{\text{IN}} + \hat{b}^\dagger_{\text{IN}}\hat{a}_{\text{IN}}).\end{aligned}$$

Adding these two equations gives the desired result,

$$\hat{a}^\dagger_{\text{OUT}}\hat{a}_{\text{OUT}} + \hat{b}^\dagger_{\text{OUT}}\hat{b}_{\text{OUT}} = \hat{a}^\dagger_{\text{IN}}\hat{a}_{\text{IN}} + \hat{b}^\dagger_{\text{IN}}\hat{b}_{\text{IN}},$$

which tells us that regardless of the joint state of the $\hat{a}_{\text{IN}}$ and $\hat{b}_{\text{IN}}$ modes, the total photon number in the output modes is the same as the total photon number in the input modes, viz., energy is conserved by this beam splitter.

(b) To prove that the beam splitter's input-output relation preserves commutator brackets is also relatively easy. We have that,

$$\begin{aligned}[\hat{a}_{\text{OUT}}, \hat{b}_{\text{OUT}}] &= [(\sqrt{\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{1-\epsilon}\,\hat{b}_{\text{IN}}), (-\sqrt{1-\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{\epsilon}\,\hat{b}_{\text{IN}})] \\ &= \epsilon[\hat{a}_{\text{IN}}, \hat{b}_{\text{IN}}] - (1-\epsilon)[\hat{b}_{\text{IN}}, \hat{a}_{\text{IN}}] = 0,\end{aligned}$$

where we have used $[\hat{a}_{\text{IN}}, \hat{b}_{\text{IN}}] = -[\hat{b}_{\text{IN}}, \hat{a}_{\text{IN}}] = 0$. Likewise, we find that,

$$\begin{aligned}[\hat{a}_{\text{OUT}}, \hat{b}^\dagger_{\text{OUT}}] &= [(\sqrt{\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{1-\epsilon}\,\hat{b}_{\text{IN}}), (-\sqrt{1-\epsilon}\,\hat{a}^\dagger_{\text{IN}} + \sqrt{\epsilon}\,\hat{b}^\dagger_{\text{IN}})] \\ &= -\sqrt{\epsilon(1-\epsilon)}[\hat{a}_{\text{IN}}, \hat{a}^\dagger_{\text{IN}}] + \sqrt{\epsilon(1-\epsilon)}[\hat{b}_{\text{IN}}, \hat{b}^\dagger_{\text{IN}}] \\ &+ \epsilon[\hat{a}_{\text{IN}}, \hat{b}^\dagger_{\text{IN}}] - (1-\epsilon)[\hat{b}_{\text{IN}}, \hat{a}^\dagger_{\text{IN}}] = 0,\end{aligned}$$

where we have used $[\hat{a}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] = -[\hat{a}^\dagger_{\rm IN}, \hat{b}_{\rm IN}]^\dagger = 0$, and $[\hat{a}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] = [\hat{b}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] = 1$. Finally we compute

$$\begin{aligned} [\hat{a}_{\rm OUT}, \hat{a}^\dagger_{\rm OUT}] &= [(\sqrt{\epsilon}\,\hat{a}_{\rm IN} + \sqrt{1-\epsilon}\,\hat{b}_{\rm IN}), (\sqrt{\epsilon}\,\hat{a}^\dagger_{\rm IN} + \sqrt{1-\epsilon}\,\hat{b}^\dagger_{\rm IN})] \\ &= \epsilon[\hat{a}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] + (1-\epsilon)[\hat{b}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] \\ &+ \sqrt{\epsilon(1-\epsilon)}\,[\hat{a}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] + \sqrt{\epsilon(1-\epsilon)}\,[\hat{b}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] = 1, \end{aligned}$$

and

$$\begin{aligned} [\hat{b}_{\rm OUT}, \hat{b}^\dagger_{\rm OUT}] &= [(-\sqrt{1-\epsilon}\,\hat{a}_{\rm IN} + \sqrt{\epsilon}\,\hat{b}_{\rm IN}), (-\sqrt{1-\epsilon}\,\hat{a}^\dagger_{\rm IN} + \sqrt{\epsilon}\,\hat{b}^\dagger_{\rm IN})] \\ &= (1-\epsilon)[\hat{a}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] + \epsilon[\hat{b}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] \\ &- \sqrt{\epsilon(1-\epsilon)}\,[\hat{a}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] - \sqrt{\epsilon(1-\epsilon)}\,[\hat{b}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] = 1. \end{aligned}$$

Note that commutator-bracket preservation is important because it means that no additional quantum noise is needed to ensure that the free-field Heisenberg uncertainty principle that applies to the input modes, $\hat{a}_{\rm IN}$ and $\hat{b}_{\rm IN}$, also applies to the output modes, $\hat{a}_{\rm IN}$ and $\hat{b}_{\rm IN}$.

(c) Characteristic functions make it easy to derive the state transformations that are produced by quantum linear systems. We have that,

$$\begin{aligned} -\zeta_a^*\hat{a}_{\rm OUT} - \zeta_b^*\hat{b}_{\rm OUT} &= -\zeta_a^*(\sqrt{\epsilon}\,\hat{a}_{\rm IN} + \sqrt{1-\epsilon}\,\hat{b}_{\rm IN}) - \zeta_b^*(-\sqrt{1-\epsilon}\,\hat{a}_{\rm IN} + \sqrt{\epsilon}\,\hat{b}_{\rm IN}) \\ &= -\zeta_a'^*\hat{a}_{\rm IN} - \zeta_b'^*\hat{b}_{\rm IN}, \end{aligned}$$

and

$$\begin{aligned} \zeta_a\hat{a}^\dagger_{\rm OUT} + \zeta_b\hat{b}^\dagger_{\rm OUT} &= \zeta_a(\sqrt{\epsilon}\,\hat{a}^\dagger_{\rm IN} + \sqrt{1-\epsilon}\,\hat{b}^\dagger_{\rm IN}) + \zeta_b(-\sqrt{1-\epsilon}\,\hat{a}^\dagger_{\rm IN} + \sqrt{\epsilon}\,\hat{b}^\dagger_{\rm IN}) \\ &= \zeta_a'\hat{a}^\dagger_{\rm IN} + \zeta_b'\hat{b}^\dagger_{\rm IN}, \end{aligned}$$

where

$$\zeta_a' \equiv \zeta_a\sqrt{\epsilon} - \zeta_b\sqrt{1-\epsilon} \quad \text{and} \quad \zeta_b' \equiv \zeta_a\sqrt{1-\epsilon} + \zeta_b\sqrt{\epsilon}.$$

It then follows that,

$$\begin{aligned} \chi_A^{\rho_{\rm OUT}}(\zeta_a^*, \zeta_b^*; \zeta_a, \zeta_b) &\equiv \langle e^{-\zeta_a^*\hat{a}_{\rm OUT} - \zeta_b^*\hat{b}_{\rm OUT}} e^{\zeta_a\hat{a}^\dagger_{\rm OUT} + \zeta_b\hat{b}^\dagger_{\rm OUT}}\rangle \\ &= \langle e^{-\zeta_a'^*\hat{a}_{\rm IN} - \zeta_b'^*\hat{b}_{\rm IN}} e^{\zeta_a'\hat{a}^\dagger_{\rm IN} + \zeta_b'\hat{b}^\dagger_{\rm IN}}\rangle \\ &= \chi_A^{\rho_{\rm IN}}(\zeta_a'^*, \zeta_b'^*; \zeta_a', \zeta_b'), \end{aligned}$$

where angle brackets denote quantum averaging, i.e., multiplication by the appropriate density operator and taking the trace.

(d) From Problem 5.3(a) we have that,

$$\chi_A^{\rho_{\rm IN}}(\zeta_a'^*, \zeta_b'^*; \zeta_a', \zeta_b') = e^{\zeta_a'\alpha_{\rm IN}^* - \zeta_a'^*\alpha_{\rm IN} - |\zeta_a'|^2} e^{\zeta_b'\beta_{\rm IN}^* - \zeta_b'^*\beta_{\rm IN} - |\zeta_b'|^2}.$$

Substituting in for ζ_a' and ζ_b' we then get,

$$\begin{aligned} \chi_A^{\rho_{\rm OUT}}(\zeta_a^*, \zeta_b^*; \zeta_a, \zeta_b) &= e^{(\zeta_a\sqrt{\epsilon}-\zeta_b\sqrt{1-\epsilon})\alpha_{\rm IN}^* - (\zeta_a^*\sqrt{\epsilon}-\zeta_b^*\sqrt{1-\epsilon})\alpha_{\rm IN} - |\zeta_a\sqrt{\epsilon}-\zeta_b\sqrt{1-\epsilon}|^2} \\ &\times\; e^{(\zeta_a\sqrt{1-\epsilon}+\zeta_b\sqrt{\epsilon})\beta_{\rm IN}^* - (\zeta_a\sqrt{1-\epsilon}+\zeta_b\sqrt{\epsilon})^*\beta_{\rm IN} - |\zeta_a\sqrt{1-\epsilon}+\zeta_b\sqrt{\epsilon}|^2} \\ &= e^{\zeta_a\alpha_{\rm OUT}^* - \zeta_a^*\alpha_{\rm OUT} - |\zeta_a|^2} e^{\zeta_b\beta_{\rm IN}^* - \zeta_b^*\beta_{\rm OUT} - |\zeta_b|^2}, \end{aligned}$$

where

$$\begin{aligned} \alpha_{\rm OUT} &\equiv \sqrt{\epsilon}\,\alpha_{\rm IN} + \sqrt{1-\epsilon}\,\beta_{\rm IN}, \\ \beta_{\rm OUT} &\equiv -\sqrt{1-\epsilon}\,\alpha_{\rm IN} + \sqrt{\epsilon}\,\beta_{\rm IN}. \end{aligned}$$

This anti-normally ordered characteristic function is, by the result of Problem 5.3(a), that of the two-mode coherent state $|\alpha_{\rm OUT}\rangle_{\rm OUT}|\beta_{\rm OUT}\rangle_{\rm OUT}$, QED.

Problem 6.2

Here we shall develop a moment-generating function approach to the quantum statistics of single-mode direct detection.

(a) We have that,

$$M_N(s) \equiv \sum_{n=0}^{\infty} e^{sn}\,\Pr(N=n) = \sum_{n=0}^{\infty} e^{sn}\langle n|\hat{\rho}|n\rangle, \quad \text{for } s \text{ real}, \tag{1}$$

and

$$Q_N(\lambda) \equiv \sum_{n=0}^{\infty} (1-\lambda)^n \langle n|\hat{\rho}|n\rangle, \quad \text{for } \lambda \text{ real}. \tag{2}$$

Thus, we see that

$$Q_N(\lambda) = M_N(s)|_{s=\ln(1-\lambda)} \quad \text{and} \quad M_N(s) = Q_N(\lambda)|_{\lambda=1-e^s}.$$

(b) Straightforward differentiation gives us,

$$\frac{d^k[(1-\lambda)^n]}{d\lambda^k} = \begin{cases} (-1)^k n(n-1)(n-2)\cdots(n-k+1)(1-\lambda)^{n-k}, & \text{for } n \ge k = 1,2,3,\ldots, \\ 0, & \text{for } k > n = 0,1,2,\ldots \end{cases}$$

Substituting this result into Eq. (2) and setting $\lambda = 0$ we obtain,

$$\left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0} = \sum_{n=k}^{\infty}(-1)^k n(n-1)(n-2)\cdots(n-k+1)\langle n|\hat{\rho}|n\rangle \quad \text{for } k = 1,2,3\ldots \tag{3}$$

Now, k-repeated applications of the annilhilation operator to the number ket $|n\rangle$ yields

$$\hat{a}^k|n\rangle = \begin{cases} \sqrt{n(n-1)(n-2)\cdots(n-k+1)}\,|n-k\rangle, & \text{for } n \geq k = 1,2,3,\ldots, \\ 0, & \text{for } k > n = 0,1,2,\ldots, \end{cases}$$

and its adjoint relation is

$$\langle n|\hat{a}^{\dagger k} = \begin{cases} \sqrt{n(n-1)(n-2)\cdots(n-k+1)}\,\langle n-k|, & \text{for } n \geq k = 1,2,3,\ldots, \\ 0, & \text{for } k > n = 0,1,2,\ldots. \end{cases}$$

Substituting these results into Eq. (3) we get,

$$\begin{aligned}\left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0} &= \sum_{n=k}^{\infty}(-1)^k\langle n|\hat{\rho}|n\rangle\langle n|\hat{a}^{\dagger k}\hat{a}^k|n\rangle \\ &= \sum_{n=k}^{\infty}(-1)^k\langle n|\hat{\rho}\hat{a}^{\dagger k}\hat{a}^k|n\rangle = (-1)^k\text{tr}(\hat{\rho}\hat{a}^{\dagger k}\hat{a}^k) \\ &= (-1)^k\langle\hat{a}^{\dagger k}\hat{a}^k\rangle,\end{aligned}$$

where the second equality follows from

$$\hat{a}^{\dagger k}\hat{a}^k|n\rangle = n(n-1)(n-2)\cdots(n-k+1)|n\rangle, \quad \text{for } n \geq k, \tag{4}$$

the third equality follows from the completness of the number kets, and the last equality follows from Problem 3.2(c).

(c) Here we assume that the field is in the mth number state, $|m\rangle$. From Eq. (4) we immediately see that,

$$\langle m|\hat{a}^{\dagger k}\hat{a}^k|m\rangle = \begin{cases} m(m-1)(m-2)\cdots(m-k+1), & \text{for } m \geq k \\ 0, & \text{for } k > m. \end{cases}$$

Using the Taylor series,

$$Q_N(\lambda) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0}\right)\lambda^k$$

we then get,

$$Q_N(\lambda) = \sum_{k=0}^{m} (-\lambda)^k \left(\begin{array}{c} m \\ k \end{array} \right).$$

From part (a) we now have,

$$M_N(s) = Q_N(\lambda)|_{\lambda=1-e^s} = \sum_{k=0}^{m} (e^s - 1)^k \left(\begin{array}{c} m \\ k \end{array} \right) = e^{sm},$$

where the last equality follows from the binomial theorem,

$$\sum_{k=0}^{m} p^k q^{m-k} \left(\begin{array}{c} m \\ k \end{array} \right) = (p+q)^m,$$

with $p \equiv e^s - 1$ and $q \equiv 1$. We see that our result for $M_N(s)$ thus derived is correct, because when the field is in the state $|m\rangle$ we have $\langle n|\hat{\rho}|n\rangle = \delta_{nm}$, whence $M_N(s) = e^{sm}$ from Eq. (1).

(d) Now we are given that the field is in the coherent state $|\alpha\rangle$. In this case it is trivial to find the factorial moments, because repeated application of the coherent-state eigenvector/eigenvalue relation gives,

$$\hat{a}^k|\alpha\rangle = \alpha^k|\alpha\rangle,$$

and the adjoint of this equation is,

$$\langle\alpha|\hat{a}^{\dagger k} = \alpha^{*k}\langle\alpha|.$$

Taking the inner product of these equations gives

$$\langle\hat{a}^{\dagger k}\hat{a}^k\rangle = |\alpha|^{2k}.$$

Once again employing the Taylor series for $Q_N(\lambda)$, we find that

$$Q_N(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}|\alpha|^{2k} = e^{-\lambda|\alpha|^2}.$$

From part (a) we now have,

$$M_N(s) = Q_N(\lambda)|_{\lambda=1-e^s} = \exp[|\alpha|^2(e^s - 1)]. \tag{5}$$

We know that

$$\Pr(N = n) = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots,$$

when the number operator is measured on the coherent-state field $|\alpha\rangle$. The moment-generating function of this Poisson distribution is easily found to be given by the second equality in Eq. (5): QED.

Problem 6.3

Here we shall examine a quantum photodetection model for single-mode direct detection with sub-unity quantum efficiency.

(a) This is a straightforward calculation. Using the definition of $\hat{a}'$ we have that

$$\langle \hat{a}'^{\dagger k}\hat{a}'^{k}\rangle = \langle (\sqrt{\eta}\hat{a}^\dagger + \sqrt{1-\eta}\hat{a}_\eta^\dagger)^k(\sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{a}_\eta)^k\rangle.$$

The signal field mode, $\hat{a}$, is in an arbitrary state, but the mode $\hat{a}_\eta$ is in its vacuum state for which $\hat{a}_\eta^m|0\rangle_\eta = 0$ and ${}_\eta\langle 0|\hat{a}_\eta^{\dagger m} = 0$ for all $m \geq 1$. Thus, because the factorial moment is a normally-ordered form, we find that the *only* term that survives the averaging is the term that contains *no* $\hat{a}_\eta^\dagger$ or $\hat{a}_\eta$ terms, viz.,

$$\langle \hat{a}'^{\dagger k}\hat{a}'^{k}\rangle = \eta^k\langle \hat{a}^{\dagger k}\hat{a}^k\rangle.$$

(b) Using the Taylor series for $Q_{N'}(\lambda)$, as in Problem 6.2, we obtain

$$Q_{N'}(\lambda) = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\eta^k\langle \hat{a}^{\dagger k}\hat{a}^k\rangle = Q_N(\eta\lambda),$$

for $Q_N(\lambda)$ as defined in Eq. (2), where the last equality makes use of the Taylor series for $Q_N(\lambda)$.

(c) This part is trivial. Using the results of Problem 6.2(a) and 6.3(b) we have that,

$$M_{N'}(s) = Q_{N'}(\lambda)|_{\lambda=1-e^s} = Q_N(\eta\lambda)|_{\lambda=1-e^s}.$$

Another result from Problem 6.2(a) yields,

$$Q_{N'}(1-e^s) = M_{N'}(s),$$

whence

$$M_{N'}(s) = Q_N[\eta(1-e^s)] = M_N\{\ln[1-\eta(1-e^s)]\}, \tag{6}$$

by yet another application of Problem 6.2(a).

(d) We are trying to prove that Eq. (6) is equivalent to,

$$M_{N'}(s) = \sum_{n=0}^{\infty}e^{sn}\left[\sum_{k=n}^{\infty}\binom{k}{n}\eta^n(1-\eta)^{k-n}\langle k|\hat{\rho}|k\rangle\right]. \tag{7}$$

We'll prove this assertion by assuming that Eq. (7) is correct and showing that Eq. (6) follows therefrom. Interchanging the orders of summation in Eq. (7) we have that,

$$\begin{aligned} M_{N'}(s) &= \sum_{k=0}^{\infty}\left[\sum_{n=0}^{k}\binom{k}{n}e^{sn}\eta^n(1-\eta)^{k-n}\right]\langle k|\hat{\rho}|k\rangle \\ &= \sum_{k=0}^{\infty}[1-\eta(1-e^s)]^k\langle k|\hat{\rho}|k\rangle = M_N\{\ln[1-\eta(1-e^s)]\}, \end{aligned}$$

where the second equality follows from the binomial theorem and the last equality follows from Problem 6.2(a): QED.

(e) Because

$$M_{N'}(s) = \sum_{n=0}^{\infty} e^{sn}\,\Pr(N' = n),$$

by definition, the result of (d) immediately gives us that

$$\Pr(N' = n) = \sum_{k=n}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} \eta^n (1-\eta)^{k-n} \langle k|\hat{\rho}|k\rangle. \tag{8}$$

Equation (8) has the following interpretation. An ideal ($\eta = 1$) photon counter when illuminated by the field in state $\hat{\rho}$ will register k counts with probability,

$$\Pr(N = k) = \langle k|\hat{\rho}|k\rangle.$$

A detector with quantum efficiency $\eta < 1$ will randomly miss a count—that the ideal detector would have made—with probability $1 - \eta$, i.e., the quantum-efficiency-η detector's counts are those of a unity-quantum-efficiency detector subjected to a Bernoulli deletion process. In other words, the conditional probability that n counts will be registered by the quantum-efficiency-η device, given that k counts are registered by a unity-quantum-efficiency device, is,

$$\Pr(\, N' = n \mid N = k\,) = \begin{pmatrix} k \\ n \end{pmatrix} \eta^n (1-\eta)^{k-n}, \quad \text{for } 0 \le n \le k.$$

Problem 6.4

Here we shall continue our investigation of quantum linear transformations by treating the single-frequency quantum theory of the degenerate parametric amplifier (DPA).

(a) Commutator preservation is easily demonstrated. We have that,

$$\begin{aligned} [\hat{a}_{\text{OUT}}, \hat{a}_{\text{OUT}}] &= [(\mu\hat{a}_{\text{IN}} + \nu\hat{a}_{\text{IN}}^\dagger), (\mu^*\hat{a}_{\text{IN}}^\dagger + \nu^*\hat{a}_{\text{IN}})] \\ &= |\mu|^2[\hat{a}_{\text{IN}}, \hat{a}_{\text{IN}}^\dagger] + |\nu|^2[\hat{a}_{\text{IN}}^\dagger, \hat{a}_{\text{IN}}] = |\mu|^2 - |\nu|^2 = 1, \end{aligned}$$

where we have used $[\hat{a}_{\text{IN}}, \hat{a}_{\text{IN}}^\dagger] = -[\hat{a}_{\text{IN}}^\dagger, \hat{a}_{\text{IN}}] = 1$.

(b) There is really no work to be done here. From Problem 5.3(a) we have that,

$$\chi_W^{\rho_{\text{IN}}}(\zeta^*, \zeta) = e^{\zeta\alpha_{\text{IN}}^* - \zeta^*\alpha_{\text{IN}} - |\zeta|^2/2}.$$

(c) Here we proceed along the lines used in Problem 6.1(c). We have that,

$$\begin{aligned}\chi_W^{\rho_{\rm OUT}}(\zeta^*,\zeta) &= \langle e^{-\zeta^*\hat{a}_{\rm OUT}+\zeta\hat{a}_{\rm OUT}^\dagger}\rangle = \langle e^{-\zeta^*(\mu\hat{a}_{\rm IN}+\nu\hat{a}_{\rm IN}^\dagger)+\zeta(\mu^*\hat{a}_{\rm IN}^\dagger+\nu^*\hat{a}_{\rm IN})}\rangle \\ &= \langle e^{-\zeta'^*\hat{a}_{\rm IN}+\zeta'\hat{a}_{\rm IN}^\dagger}\rangle = \chi_W^{\rho_{\rm IN}}(\zeta'^*,\zeta'),\end{aligned}$$

where $\zeta' \equiv -\zeta^*\nu + \zeta\mu^*$, and angle brackets denote quantum averaging, i.e., multiplication by the appropriate density operator and taking the trace.

(d) Let $a_{{\rm OUT}_1}$ and $a_{{\rm OUT}_2}$ denote the classical outcomes of the $\hat{a}_{{\rm OUT}_1}$ and $\hat{a}_{{\rm OUT}_2}$ measurements. The classical characteristic functions of these measurement outcomes can be found as follows:

$$\begin{aligned}M_{a_{{\rm OUT}_1}}(jv) &\equiv E(e^{jva_{{\rm OUT}_1}}) = \langle e^{jv\hat{a}_{{\rm OUT}_1}}\rangle = \langle e^{(jv/2)\hat{a}_{\rm OUT}+(jv/2)\hat{a}_{\rm OUT}^\dagger}\rangle \\ &= \chi_W^{\rho_{\rm OUT}}(-jv/2, jv/2) = \chi_W^{\rho_{\rm IN}}(-(jv/2)(\mu+\nu),(jv/2)(\mu+\nu))\end{aligned}$$

and,

$$\begin{aligned}M_{a_{{\rm OUT}_2}}(jv) &\equiv E(e^{jva_{{\rm OUT}_2}}) = \langle e^{jv\hat{a}_{{\rm OUT}_2}}\rangle = \langle e^{(v/2)\hat{a}_{\rm OUT}-(v/2)\hat{a}_{\rm OUT}^\dagger}\rangle \\ &= \chi_W^{\rho_{\rm OUT}}(-v/2, -v/2) = \chi_W^{\rho_{\rm IN}}(-(v/2)(\mu-\nu),-(v/2)(\mu-\nu)),\end{aligned}$$

where we have used the fact that μ and ν are real valued. Now, using the Wigner characteristic function of the input-mode coherent state we get our final characteristic-function results:

$$\begin{aligned}M_{a_{{\rm OUT}_1}}(jv) &= e^{(jv/2)(\mu+\nu)\alpha_{\rm IN}+(jv/2)(\mu+\nu)\alpha_{\rm IN}^*-(v^2/8)(\mu+\nu)^2} \\ &= e^{jv(\mu+\nu)\alpha_{{\rm IN}_1}-(v^2/8)(\mu+\nu)^2} \\ M_{a_{{\rm OUT}_2}}(jv) &= e^{(v/2)(\mu-\nu)\alpha_{\rm IN}-(v/2)(\mu-\nu)\alpha_{\rm IN}^*-(v^2/8)(\mu-\nu)^2} \\ &= e^{jv(\mu-\nu)\alpha_{{\rm IN}_2}-(v^2/8)(\mu-\nu)^2},\end{aligned}$$

where $\alpha_{{\rm IN}_1}$ and $\alpha_{{\rm IN}_2}$ are the real and imaginary parts of $\alpha_{\rm IN}$, respectively. By inspection, we see that these are the characteristic functions of classical Gaussian random variables. In particular, $a_{{\rm OUT}_1}$ is Gaussian distributed with mean value $(\mu+\nu)\alpha_{{\rm IN}_1}$ and variance $(\mu+\nu)^2/4$, and $a_{{\rm OUT}_2}$ is Gaussian distributed with mean value $(\mu-\nu)\alpha_{{\rm IN}_2}$ and variance $(\mu-\nu)^2/4$. Note that the mean values are in accord with what we would find directly by taking the quadrature components of the quantum average of the DPA's input-output relation, viz.,

$$\langle\hat{a}_{\rm OUT}\rangle = \mu\langle\hat{a}_{\rm IN}\rangle + \nu\langle\hat{a}_{\rm IN}^\dagger\rangle = \mu\langle\hat{a}_{\rm IN}\rangle + \nu\langle\hat{a}_{\rm IN}\rangle^* = \mu\alpha_{\rm IN} + \nu\alpha_{\rm IN}^*.$$

Also note that the variances satisfy the Heisenberg uncertainty principle with equality,

$$\langle \Delta \hat{a}^2_{\mathrm{OUT}_1} \rangle \langle \Delta \hat{a}^2_{\mathrm{OUT}_2} \rangle = \frac{(\mu+\nu)^2(\mu-\nu)^2}{16} = \frac{(\mu^2-\nu^2)^2}{16} = \frac{1}{16},$$

where we have used the fact that μ and ν are real valued. This is as it should be, because we showed in class that the Bogoliubov transformation with μ and ν real produces squeezed states.