Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 4 Solutions
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Problem 4.1
Here we shall show that the creation operator, $\hat{a}^\dagger$, does *not* have any non-zero eigenkets. Suppose that

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle, \tag{1}$$

for β a complex number. Using the completeness of the number kets,

$$\hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|,$$

we have that

$$|\beta\rangle = \hat{I}|\beta\rangle = \sum_{n=0}^{\infty}\langle n|\beta\rangle|n\rangle = \sum_{n=0}^{\infty} b_n|n\rangle,$$

with the obvious definition of the $\{b_n\}$. Substituting this expansion into Eq. (1) then gives,

$$\hat{a}^\dagger|\beta\rangle = \sum_{n=0}^{\infty} b_n\hat{a}^\dagger|n\rangle = \sum_{n=0}^{\infty} b_n\sqrt{n+1}|n+1\rangle = \beta|\beta\rangle = \sum_{n=0}^{\infty}\beta b_n|n\rangle. \tag{2}$$

Now, because $|n\rangle$ and $|m\rangle$ are orthogonal for $n \neq m$, it must be that the coefficients of the same number ket on each side of Eq. (2) are equal. For $n = 0$ this gives $0 = \beta b_0$. Unless $\beta = 0$—a case we will treat momentarily—it follows that $b_0 = 0$. For $n > 0$, equating the corresponding number-ket coefficients on both sides of Eq. (2) gives,

$$b_{n-1}\sqrt{n} = \beta b_n \quad \longrightarrow \quad b_n = \frac{b_0\sqrt{n!}}{\beta^n},$$

assuming $\beta \neq 0$. But, if $\beta \neq 0$ we already know that $b_0 = 0$. Thus, $\hat{a}^\dagger$ has no non-zero eigenkets with non-zero eigenvalues.

Maybe there is a non-zero eigenket with zero eigenvalue? Suppose $|\beta_0\rangle$ is such a state, viz., $|\psi_0\rangle \equiv \hat{a}^\dagger|\beta_0\rangle = 0$. Then the squared length of $|\psi_0\rangle$ must be zero, i.e.,

$$0 = \langle\psi_0|\psi_0\rangle = \langle\beta_0|\hat{a}\hat{a}^\dagger|\beta_0\rangle = 0 = \langle\beta_0|\hat{a}^\dagger\hat{a}|\beta_0\rangle + \langle\beta_0|\beta_0\rangle \geq \langle\beta_0|\beta_0\rangle \geq 0,$$

where the last equality used $[\hat{a}, \hat{a}^\dagger] = 1$. This result shows that $|\beta_0\rangle$ has zero length, proving that $\hat{a}^\dagger$ has no non-zero eigenket with zero eigenvalue.

Problem 4.2
Here we shall work out some properties of the coherent states.

(a) We showed in class that the coherent state,

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-|\alpha|^2/2)|n\rangle,$$

is an eigenket of the annihilation operator with eigenvalue α, viz., $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, for any complex number α. This is a remarkable result, because $\hat{a}$ is not an Hermitian operator, moreover its real and imaginary parts, $\hat{a}_1 \equiv (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{a}_2 \equiv (\hat{a} - \hat{a}^\dagger)/2j$, don't commute. Anyway, the result we are seeking in this part of the problem is easy to get. We start out as follows:

$$\begin{aligned}
\langle\alpha|\beta\rangle &= \left(\sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \exp(-|\alpha|^2/2)\langle n|\right)\left(\sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} \exp(-|\beta|^2/2)|m\rangle\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \exp(-|\alpha|^2/2)(\langle n|m\rangle)\frac{\beta^m}{\sqrt{m!}} \exp(-|\beta|^2/2) \\
&= \left(\sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!}\right) \exp[-(|\alpha|^2 + |\beta|^2)/2],
\end{aligned}$$

where we have used the orthonormality of the number kets to obtain the last equality. Now, using the power series for the exponential function we get the inner product we were seeking:

$$\langle\alpha|\beta\rangle = \exp[\alpha^*\beta - (|\alpha|^2 + |\beta|^2)/2].$$

We see from this result that the coherent states are normalized to unit length, $\langle\alpha|\alpha\rangle = 1$, but they are not orthogonal, i.e., $\alpha \neq \beta$ does not imply $\langle\alpha|\beta\rangle = 0$.

(b) Any operator is characterized by its matrix elements in a complete orthonormal basis. Using the number kets as that basis, we have that

$$\begin{aligned}
\langle n|\left(\int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|\right)|m\rangle &= \int \frac{d^2\alpha}{\pi}\,(\langle n|\alpha\rangle)(\langle\alpha|m\rangle) \\
&= \int \frac{d^2\alpha}{\pi}\,\frac{\alpha^{*n}\alpha^m}{\sqrt{n!m!}} \exp[-|\alpha|^2] \\
&= \int_0^{\infty} dz\, z \int_0^{2\pi} \frac{d\phi}{\pi}\,\frac{z^{n+m}e^{j(m-n)\phi}}{\sqrt{n!m!}} \exp(-z^2),
\end{aligned}$$

where $ze^{j\phi}$ is the polar-coordinate form of $\alpha = \alpha_1 + j\alpha_2$. The ϕ integral is easily shown to be zero for $n \neq m$, and when $n = m$ we have that

$$\begin{aligned} \langle n| \left(\int \frac{d^2\alpha}{\pi} \, |\alpha\rangle\langle\alpha| \right) |n\rangle &= \int_0^\infty dz \, \frac{2z^{2n+1}}{n!} \, \exp(-z^2) \\ &= \int_0^\infty dy \, \frac{y^n}{n!} e^{-y} = 1, \end{aligned}$$

where the penultimate equality used the change of variables $y \equiv z^2$, and the final equality follows from the factorial integral given in Problem 1.5 (a). These matrix elements coincide with those of the identity operator,

$$\langle n|\hat{I}|m\rangle = \delta_{nm},$$

thus we have that

$$\hat{I} = \int \frac{d^2\alpha}{\pi} \, |\alpha\rangle\langle\alpha|,$$

QED.

Comment: Because the coherent states are not an orthogonal set, but their outer-product sum resolves the identity as just shown, they form an overcomplete set, cf. Problem 1.8. It follows that any state $|\psi\rangle$ has a coherent-state expansion of the form,

$$|\psi\rangle = \hat{I}|\psi\rangle = \int \frac{d^2\alpha}{\pi} \, \psi(\alpha)|\alpha\rangle,$$

with $\psi(\alpha) \equiv \langle\alpha|\psi\rangle$.

(c) All of these results are easily obtained from the overcompleteness relation plus the eigenvalue/eigenket property of the coherent states. We have that,

$$\begin{aligned} \hat{a} = \hat{a}\hat{I} &= \int \frac{d^2\alpha}{\pi} \, (\hat{a}|\alpha\rangle)\langle\alpha| \\ &= \int \frac{d^2\alpha}{\pi} \, \alpha|\alpha\rangle\langle\alpha|. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} \hat{a}^\dagger = \hat{I}\hat{a}^\dagger &= \int \frac{d^2\alpha}{\pi} \, |\alpha\rangle(\langle\alpha|\hat{a}^\dagger) \\ &= \int \frac{d^2\alpha}{\pi} \, \alpha^*|\alpha\rangle\langle\alpha|. \end{aligned}$$

Continuing in this manner, we next show that

$$\begin{aligned}
\hat{a}\hat{a}^\dagger = \hat{a}\hat{I}\hat{a}^\dagger &= \int \frac{d^2\alpha}{\pi}\,(\hat{a}|\alpha\rangle)(\langle\alpha|\hat{a}^\dagger) \\
&= \int \frac{d^2\alpha}{\pi}\,|\alpha|^2|\alpha\rangle\langle\alpha|.
\end{aligned}$$

For the final result, we use the previous work plus the commutator, $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$, to show that

$$\begin{aligned}
\hat{a}^\dagger\hat{a} &= \hat{a}\hat{a}^\dagger - \hat{I} \\
&= \int \frac{d^2\alpha}{\pi}\,(|\alpha|^2 - 1)|\alpha\rangle\langle\alpha|.
\end{aligned}$$

Problem 4.3

Here we will explore the phase behavior of the quantum harmonic oscillator whose photon annihilation operator is $\hat{a}$. The Susskind-Glogower phase operator $\widehat{e^{j\phi}}$ associated with $\hat{a}$ is defined as follows

$$\widehat{e^{j\phi}} \equiv (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}.$$

(Note that the "widehat" symbol is used to indicate that this is *not* the exponentiation of j times an Hermitian operator $\hat{\phi}$.)

(a) To find the number-ket representation of $\widehat{e^{j\phi}}$, we make use of

$$\hat{a} = \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n|,$$

and

$$\hat{a}\hat{a}^\dagger = \sum_{m=0}^{\infty} (m+1)|m\rangle\langle m|.$$

Then, from the useful fact, we get

$$\widehat{e^{j\phi}} = \sum_{m=0}^{\infty} (m+1)^{-1/2}|m\rangle\langle m| \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n| = \sum_{n=1}^{\infty} |n-1\rangle\langle n|$$

where the second equality follows from

$$\langle m|n-1\rangle = \begin{cases} 1, & \text{for } m = n-1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) To verify that

$$|e^{j\phi}\rangle \equiv \sum_{n=0}^{\infty} e^{jn\phi}|n\rangle,$$

is an eigenket of $\widehat{e^{j\phi}}$, we use the result of (a) and find

$$\begin{aligned}
\widehat{e^{j\phi}}|e^{j\phi}\rangle &= \sum_{m=1}^{\infty} |m-1\rangle\langle m| \sum_{n=0}^{\infty} e^{jn\phi}|n\rangle \\
&= \sum_{m=1}^{\infty} e^{jm\phi}|m-1\rangle \\
&= e^{j\phi}\sum_{k=0}^{\infty} e^{jk\phi}|k\rangle\langle k| = e^{j\phi}|e^{j\phi}\rangle,
\end{aligned}$$

where the $\{|n\rangle\}$, $\{|m\rangle\}$, and $\{|k\rangle\}$ are number kets, the second equality follows from number-ket orthonormality, and the third equality follows from $k \equiv m-1$.

This result verifies that $|e^{j\phi}\rangle$ is an eigenket of $\widehat{e^{j\phi}}$ whose associated eigenvalue is $e^{j\phi}$. Note that $\widehat{e^{j\phi}}$ is *not* Hermitian, so it is somewhat surprising that it has eigenkets. Given that it does, however, it is not surprising that they turned out to be complex valued.

(c) To show that $\{\,|e^{j\phi}\rangle : -\pi \le \phi \le \pi\,\}$ resolves the identity, i.e., to prove that

$$\hat{I} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, |e^{j\phi}\rangle\langle e^{j\phi}|,$$

it suffices to verify that

$$\langle n| \left(\int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, |e^{j\phi}\rangle\langle e^{j\phi}|\right)|m\rangle = \langle n|\hat{I}|m\rangle = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

Using the number-ket expansions for $|e^{j\phi}\rangle$ and $\langle e^{j\phi}|$ we get

$$\langle n| \left(\int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, |e^{j\phi}\rangle\langle e^{j\phi}|\right)|m\rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, e^{j(n-m)\phi} = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

as desired.

Because

$$\hat{\Pi}(\phi) \equiv \frac{|e^{j\phi}\rangle\langle e^{j\phi}|}{2\pi}$$

is a positive constant times a projector, it is both Hermitian and positive semidefinite. Thus, because $\hat{\Pi}(\phi)$ resolves the identity, it is a probability operator-valued measurement (POVM). Indeed the outcome from measuring this POVM is a 2π-rad phase observation, $-\pi \le \phi < \pi$, on the quantum harmonic oscillator whose annihilation operator is $\hat{a}$.

Problem 4.4
Here we shall develop a little commutator calculus that will be needed in the next problem.

(a) We start with

$$\left[\hat{a}_1, \hat{a}_2^2\right] \equiv \hat{a}_1\hat{a}_2^2 - \hat{a}_2^2\hat{a}_1 = \left(\left[\hat{a}_1, \hat{a}_2\right] + \hat{a}_2\hat{a}_1\right)\hat{a}_2 - \hat{a}_2^2\hat{a}_1,$$

and then employ the commutator $[\hat{a}_1, \hat{a}_2] = j/2$ to get

$$\begin{aligned}\left[\hat{a}_1, \hat{a}_2^2\right] &= \left(j/2 + \hat{a}_2\hat{a}_1\right)\hat{a}_2 - \hat{a}_2^2\hat{a}_1 = \hat{a}_2\left(j/2 + \hat{a}_1\hat{a}_2 - \hat{a}_2\hat{a}_1\right) \\ &= \hat{a}_2\left(j/2 + [\hat{a}_1, \hat{a}_2]\right) = j\hat{a}_2.\end{aligned}$$

Now let us start with

$$\left[\hat{a}_1, \hat{a}_2^{k+1}\right] = \left(\hat{a}_1\hat{a}_2^{k+1} - \hat{a}_2^{k+1}\hat{a}_1\right) = \left(\left[\hat{a}_1, \hat{a}_2^k\right] + \hat{a}_2^k\hat{a}_1\right)\hat{a}_2 - \hat{a}_2^{k+1}\hat{a}_1,$$

and apply the assumed commutator $\left[\hat{a}_1, \hat{a}_2^k\right] = jk\hat{a}_2^{k-1}/2$ to get

$$\begin{aligned}\left[\hat{a}_1, \hat{a}_2^{k+1}\right] &= \left(jk\hat{a}_2^{k-1}/2 + \hat{a}_2^k\hat{a}_1\right)\hat{a}_2 - \hat{a}_2^{k+1}\hat{a}_1 \\ &= jk\hat{a}_2^k/2 + \hat{a}_2^k\left([\hat{a}_1, \hat{a}_2]\right) \\ &= jk\hat{a}_2^k/2 + j\hat{a}_2^k/2 = j(k+1)\hat{a}_2^k/2,\end{aligned}$$

QED for the induction.

(b) It's déjà vu all over again! We start with

$$\left[\hat{a}_2, \hat{a}_1^2\right] \equiv \hat{a}_2\hat{a}_1^2 - \hat{a}_1^2\hat{a}_2 = \left(\left[\hat{a}_2, \hat{a}_1\right] + \hat{a}_1\hat{a}_2\right)\hat{a}_1 - \hat{a}_1^2\hat{a}_2,$$

and then employ the commutator $[\hat{a}_2, \hat{a}_1] = -j/2$ to get

$$\begin{aligned}\left[\hat{a}_2, \hat{a}_1^2\right] &= \left(-j/2 + \hat{a}_1\hat{a}_2\right)\hat{a}_1 - \hat{a}_1^2\hat{a}_2 = \hat{a}_1\left(-j/2 + \hat{a}_2\hat{a}_1 - \hat{a}_1\hat{a}_2\right) \\ &= \hat{a}_1\left(-j/2 + [\hat{a}_2, \hat{a}_1]\right) = -j\hat{a}_1.\end{aligned}$$

Now let us start with

$$\left[\hat{a}_2, \hat{a}_1^{k+1}\right] = \left(\hat{a}_2\hat{a}_1^{k+1} - \hat{a}_1^{k+1}\hat{a}_2\right) = \left(\left[\hat{a}_2, \hat{a}_1^k\right] + \hat{a}_1^k\hat{a}_2\right)\hat{a}_1 - \hat{a}_1^{k+1}\hat{a}_2,$$

and apply the assumed commutator $\left[\hat{a}_2, \hat{a}_1^k\right] = -jk\hat{a}_1^{k-1}/2$ to get

$$\begin{aligned}\left[\hat{a}_2, \hat{a}_1^{k+1}\right] &= \left(-jk\hat{a}_1^{k-1}/2 + \hat{a}_1^k\hat{a}_2\right)\hat{a}_1 - \hat{a}_1^{k+1}\hat{a}_2 \\ &= -jk\hat{a}_1^k/2 + \hat{a}_1^k\left([\hat{a}_2, \hat{a}_1]\right) \\ &= -jk\hat{a}_1^k/2 - j\hat{a}_1^k/2 = -j(k+1)\hat{a}_1^k/2,\end{aligned}$$

QED for this induction.

(c) These commutator calculations are straightforward. We have that

$$\begin{aligned}
[\hat{a}_1, G(\hat{a}_2)] &= \left[\hat{a}_1, \sum_{n=0}^{\infty} \frac{\hat{a}_2^n}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0}\right] \\
&= \sum_{n=0}^{\infty} [\hat{a}_1, \hat{a}_2^n] \frac{1}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0},
\end{aligned}$$

because we can interchange the order of the linear operations of summation and commutator evaluation. Applying the result from (a) then gives us,

$$\begin{aligned}
[\hat{a}_1, G(\hat{a}_2)] &= \sum_{n=0}^{\infty} (jn/2)\hat{a}_2^{n-1} \frac{1}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0} \\
&= (j/2)\frac{d}{d\hat{a}_2}\left(\sum_{n=0}^{\infty} \frac{\hat{a}_2^n}{n!} \left.\frac{d^n G(\alpha_2)}{d\alpha_2^n}\right|_{\alpha_2=0}\right) = (j/2)\frac{dG(\hat{a}_2)}{d\hat{a}_2}.
\end{aligned}$$

The other commutator that we are seeking follows by the same procedure:

$$\begin{aligned}
[\hat{a}_2, F(\hat{a}_1)] &= \left[\hat{a}_2, \sum_{n=0}^{\infty} \frac{\hat{a}_1^n}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0}\right] \\
&= \sum_{n=0}^{\infty} [\hat{a}_2, \hat{a}_1^n] \frac{1}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0} \\
&= -\sum_{n=0}^{\infty} (jn/2)\hat{a}_1^{n-1} \frac{1}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0} \\
&= -(j/2)\frac{d}{d\hat{a}_1}\left(\sum_{n=0}^{\infty} \frac{\hat{a}_1^n}{n!} \left.\frac{d^n F(\alpha_1)}{d\alpha_1^n}\right|_{\alpha_1=0}\right) = -(j/2)\frac{dF(\hat{a}_1)}{d\hat{a}_1}.
\end{aligned}$$

Problem 4.5

Here we shall show that the eigenkets of a quadrature operator can be found from a translation operator applied to the zero-eigenvalue eigenket.

(a) We have that

$$\begin{aligned}
\hat{a}_1 \hat{A}_1(\xi)|\alpha_1\rangle_1 &= \hat{A}_1(\xi)\hat{a}_1|\alpha_1\rangle_1 + \left[\hat{a}_1, \hat{A}_1(\xi)\right]|\alpha_1\rangle_1 \\
&= \hat{A}_1(\xi)\hat{a}_1|\alpha_1\rangle_1 + (j/2)\frac{d\hat{A}_1(\xi)}{d\hat{a}_2}|\alpha_1\rangle_1,
\end{aligned}$$

where the last equality follows from Problem 4.4(c). Now, because $\hat{a}_1|\alpha_1\rangle_1 = \alpha_1|\alpha_1\rangle_1$ and

$$\begin{aligned}
\frac{d\hat{A}_1(\xi)}{d\hat{a}_2} &= \sum_{n=0}^{\infty} \frac{(-2j\xi)^n}{n!}\frac{d\hat{a}_2^n}{d\hat{a}_2} \\
&= \sum_{n=1}^{\infty} \frac{(-2j\xi)^n}{(n-1)!}\hat{a}_2^{n-1} = (-2j\xi)\hat{A}_1(\xi),
\end{aligned}$$

we see that

$$\hat{a}_1\hat{A}_1(\xi)|\alpha_1\rangle_1 = (\alpha_1+\xi)\hat{A}_1(\xi)|\alpha_1\rangle_1,$$

i.e., $\hat{A}_1(\xi)|\alpha_1\rangle_1$ is an eigenket of $\hat{a}_1$ with eigenvalue $\alpha_1+\xi$.

(b) By definition, $\hat{A}_1(\alpha_1) = \exp(-2j\alpha_1\hat{a}_2)$. Therefore from (a) we have that $\hat{A}_1(\alpha_1)|0\rangle_1$ is an eigenket of $\hat{a}_1$ with eigenvalue α_1 if $|0\rangle_1$ is the $\hat{a}_1$ eigenket with eigenvalue zero. The length of this ket satisfies,

$$\left({}_1\langle 0|\hat{A}_1^\dagger(\alpha_1)\right)\left(\hat{A}_1(\alpha_1)|0\rangle_1\right) = {}_1\langle 0|\hat{A}_1^\dagger(\alpha_1)\hat{A}_1(\alpha_1)|0\rangle_1.$$

Moreover $\hat{A}_1^\dagger(\alpha_1) = \hat{A}_1^{-1}(\alpha_1)$, as can be verified by a tedious power-series-expansion proof that $\exp(2j\alpha_1\hat{a}_2)\exp(-2j\alpha_1\hat{a}_2) = \exp(2j\alpha_1\hat{a}_2 - 2j\alpha_1\hat{a}_2) = \hat{I}$. Thus, we get the desired result:

$$\left({}_1\langle 0|\hat{A}_1^\dagger(\alpha_1)\right)\left(\hat{A}_1(\alpha_1)|0\rangle_1\right) = {}_1\langle 0|0\rangle_1.$$

(c) This part is just a rehash of (a):

$$\begin{aligned}
\hat{a}_2\hat{A}_2(\xi)|\alpha_2\rangle_2 &= \hat{A}_2(\xi)\hat{a}_2|\alpha_2\rangle_2 + \left[\hat{a}_2, \hat{A}_2(\xi)\right]|\alpha_2\rangle_2 \\
&= \hat{A}_2(\xi)\hat{a}_2|\alpha_2\rangle_2 - (j/2)\frac{d\hat{A}_2(\xi)}{d\hat{a}_1}|\alpha_2\rangle_2 \\
&= \hat{A}_2(\xi)\alpha_2|\alpha_2\rangle_2 + \xi\hat{A}_2(\xi)|\alpha_2\rangle_2,
\end{aligned}$$

QED.

(d) This part is just a rehash of (b). By definition, $\hat{A}_2(\alpha_2) = \exp(2j\alpha_2\hat{a}_1)$. Therefore from (c) we have that $\hat{A}_2(\alpha_2)|0\rangle_2$ is an eigenket of $\hat{a}_2$ with eigenvalue α_2 if $|0\rangle_2$ is the $\hat{a}_2$ eigenket with eigenvalue zero. The length of this ket satisfies,

$$\left({}_2\langle 0|\hat{A}_2^\dagger(\alpha_2)\right)\left(\hat{A}_2(\alpha_2)|0\rangle_2\right) = {}_2\langle 0|\hat{A}_2^\dagger(\alpha_2)\hat{A}_2(\alpha_2)|0\rangle_2 = {}_2\langle 0|0\rangle_2,$$

because, $\hat{A}_2^\dagger(\alpha_2) = \hat{A}_2^{-1}(\alpha_2)$.

Problem 4.6
Here we shall continue our development of the quadrature-operator eigenkets.

(a) Suppose that the harmonic oscillator is in the $\hat{a}_1$ eigenket $|\alpha_1\rangle_1$. The average energy in this state satisfies,

$$\begin{aligned} {}_1\langle\alpha_1|\hat{H}|\alpha_1\rangle_1 &= \hbar\omega({}_1\langle\alpha_1|\hat{a}_1^2|\alpha_1\rangle_1 + {}_1\langle\alpha_1|\hat{a}_2^2|\alpha_1\rangle_1) \\ &\geq \hbar\omega({}_1\langle\alpha_1|\Delta\hat{a}_1^2|\alpha_1\rangle_1 + {}_1\langle\alpha_1|\Delta\hat{a}_2^2|\alpha_1\rangle_1), \end{aligned}$$

because mean-square values equal or exceed variances. The Heisenberg uncertainty principle then yields,

$${}_1\langle\alpha_1|\hat{H}|\alpha_1\rangle_1 \geq \hbar\omega({}_1\langle\alpha_1|\Delta\hat{a}_1^2|\alpha_1\rangle_1 + 1/16\,{}_1\langle\alpha_1|\Delta\hat{a}_1^2|\alpha_1\rangle_1) = \infty,$$

where the last equality follows from the eigenket property,

$$\hat{a}_1^k|\alpha_1\rangle_1 = \alpha_1^k|\alpha_1\rangle_1, \quad \text{for } k = 1, 2, \ldots,$$

which implies

$${}_1\langle\alpha_1|\Delta\hat{a}_1^2|\alpha_1\rangle_1 = 0.$$

Similarly, if the harmonic oscillator is in the $\hat{a}_2$ eigenket $|\alpha_2\rangle_2$, we have that,

$$\begin{aligned} {}_2\langle\alpha_2|\hat{H}|\alpha_2\rangle_2 &= \hbar\omega({}_2\langle\alpha_2|\hat{a}_1^2|\alpha_2\rangle_2 + {}_2\langle\alpha_2|\hat{a}_2^2|\alpha_2\rangle_2) \\ &\geq \hbar\omega({}_2\langle\alpha_2|\Delta\hat{a}_1^2|\alpha_2\rangle_2 + {}_2\langle\alpha_2|\Delta\hat{a}_2^2|\alpha_2\rangle_2) \\ &\geq \hbar\omega(1/16\,{}_2\langle\alpha_2|\Delta\hat{a}_2^2|\alpha_2\rangle_2 + {}_2\langle\alpha_2|\Delta\hat{a}_2^2|\alpha_2\rangle_2) = \infty. \end{aligned}$$

The preceding proofs were fairly straightforward, because we used $\langle\hat{H}\rangle = \langle\psi|\hat{H}|\psi\rangle$ to calculate average energy of a state $|\psi\rangle$, rather than introducing the probability density functions for the $\hat{a}_1$ and $\hat{a}_2$ measurements, then calculate the mean-squared values of these quadrature measurements, and finally use $\langle\hat{H}\rangle = \hbar\omega\left(\langle\Delta\hat{a}_1^2\rangle + \langle\Delta\hat{a}_2^2\rangle\right)$. The problem with this latter (longer) approach arises here because we are dealing with the $\hat{a}_1$ and $\hat{a}_2$ eigenkets, which have infinite length. Thus the measurement statistics for these states, $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$, have to be handled somewhat differently than is the case for unit-length kets. In particular, if we measure the quantum harmonic oscillator's $\hat{a}_1$ quadrature operator when the oscillator is in a (unit-length, i.e., finite energy) state $|\psi\rangle$, then the classical probability density for the measurement outcome to be α_1 is,

$$p(\alpha_1) = |{}_1\langle\alpha_1|\psi\rangle|^2, \quad \text{for } -\infty < \alpha_1 < \infty.$$

This equation is consistent with standard probability theory because the magnitude squared on the right ensures that $p(\alpha_1) \geq 0$ for all α_1, and

$$\begin{aligned}\int_{-\infty}^{\infty} d\alpha_1\, p(\alpha_1) &= \langle\psi|\left(\int_{-\infty}^{\infty} d\alpha_1\, |\alpha_1\rangle_{11}\langle\alpha_1|\right)|\psi\rangle = \langle\psi|\hat{I}|\psi\rangle \\ &= \langle\psi|\psi\rangle = 1,\end{aligned}$$

where the second equality uses the completeness relation for the $\{|\alpha_1\rangle_1\}$, and the last equality uses the fact that $|\psi\rangle$ has unit length. The nth moment of the measurement outcome therefore satisfies,

$$\int_{-\infty}^{\infty} d\alpha_1\, \alpha_1^n p(\alpha_1) = \langle\psi|\left(\int_{-\infty}^{\infty} d\alpha_1\, \alpha_1^n|\alpha_1\rangle_{11}\langle\alpha_1|\right)|\psi\rangle = \langle\psi|\hat{a}_1^n|\psi\rangle.$$

However, if the oscillator is in the (infinite-length, hence infinite-energy) state $|\psi\rangle = |\alpha_1'\rangle_1$, then

$$p(\alpha_1) = |_1\langle\alpha_1|\psi\rangle|^2 = |\delta(\alpha_1 - \alpha_1')|^2, \quad \text{for } -\infty < \alpha_1 < \infty,$$

which is *not* an acceptable probability density for a classical random variable ($\hat{a}_1$-measurement outcome) α_1. Because of the eigenket property of $|\alpha_1\rangle$, we have that,

$$(\hat{a}_1 - \alpha_1)|\alpha_1\rangle_1 = (\alpha_1 - \alpha_1)|\alpha_1\rangle_1 = 0,$$

so that with $\langle\hat{a}_1\rangle = \alpha_1$ and with $\Delta\hat{a}_1 \equiv \hat{a}_1 - \langle\hat{a}_1\rangle$ we find that,

$$(\Delta\hat{a}_1)^2|\alpha_1\rangle_1 = 0,$$

which implies that the $\Delta\hat{a}_1$ measurement has outcome zero with probability one, in keeping with what we expect from measuring an observable when the quantum system's state is an eigenket of that observable.

(b) From Problem 4.5 we know that $|\alpha_1\rangle_1 = \hat{A}_1(\alpha_1)|0\rangle_1$. Thus,

$$_2\langle\alpha_2|\alpha_1\rangle_1 = {}_2\langle\alpha_2|\hat{A}_1(\alpha_1)|0\rangle_1.$$

Using the power series for $\hat{A}_1(\alpha_1)$, we see that

$$\begin{aligned}{}_2\langle\alpha_2|\hat{A}_1(\alpha_1) &= \sum_{n=0}^{\infty}\frac{(-2j\alpha_1)^n}{n!}{}_2\langle\alpha_2|\hat{a}_2^n \\ &= \sum_{n=0}^{\infty}\frac{(-2j\alpha_1\alpha_2)^n}{n!}{}_2\langle\alpha_2| = \exp(-2j\alpha_1\alpha_2)_2\langle\alpha_2|,\end{aligned}$$

so that we get

$$ {}_2\langle\alpha_2|\alpha_1\rangle_1 = \exp(-2j\alpha_1\alpha_2)_2\langle\alpha_2|0\rangle_1. $$

Now, from Problem 4.5 we have that

$$ {}_2\langle\alpha_2| = {}_2\langle 0|\hat{A}_2^\dagger(\alpha_2), $$

whence

$$ {}_2\langle\alpha_2|\alpha_1\rangle_1 = \exp(-2j\alpha_1\alpha_2)_2\langle 0|\hat{A}_2^\dagger(\alpha_2)|0\rangle_1. $$

To complete our derivation, we note that

$$ \hat{A}_2^\dagger(\alpha_2)|0\rangle_1 = \sum_{n=0}^{\infty}\frac{(-2j\alpha_2)^n}{n!}\hat{a}_1^n|0\rangle_1 = |0\rangle_1, $$

and so obtain the desired result, ${}_2\langle\alpha_2|\alpha_1\rangle_1 = \exp(-2j\alpha_1\alpha_2)_2\langle 0|0\rangle_1$.

(d) Start from the orthonormality relation,

$$ {}_2\langle\alpha_2'|\alpha_2\rangle_2 = \delta(\alpha_2 - \alpha_2'\,), $$

and evaluate the left-hand side via the completeness relation for the $\{|\alpha_1\rangle_1\}$, i.e.,

$$ {}_2\langle\alpha_2'|\alpha_2\rangle_2 = {}_2\langle\alpha_2'|\hat{I}|\alpha_2\rangle_2 = {}_2\langle\alpha_2'|\left(\int_{-\infty}^{\infty} d\alpha_1\,|\alpha_1\rangle_{1\,1}\langle\alpha_1|\right)|\alpha_2\rangle_2. $$

Using the result of (b) we then have that,

$$ \begin{aligned} {}_2\langle\alpha_2'|\alpha_2\rangle_2 &= \int_{-\infty}^{\infty} d\alpha_1\,{}_2\langle\alpha_2'|\alpha_1\rangle_{1\,1}\langle\alpha_1|\alpha_2\rangle_2 \\ &= |{}_2\langle 0|0\rangle_1|^2\int_{-\infty}^{\infty} d\alpha_1\,\exp[-2j(\alpha_2' - \alpha_2)\alpha_1] = |{}_2\langle 0|0\rangle_1|^2\pi\delta(\alpha_2 - \alpha_2'\,). \end{aligned} $$

With the assumption that ${}_2\langle 0|0\rangle_1$ is positive real, we get

$$ {}_2\langle 0|0\rangle_1 = \frac{1}{\sqrt{\pi}}. $$

The final result is then

$$ {}_2\langle\alpha_2|\alpha_1\rangle_1 = \frac{\exp(-2j\alpha_1\alpha_2)}{\sqrt{\pi}}. $$