Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 2

Fall 2016

Issued: Thursday, September 15, 2016 **Due:** Thursday, September 22, 2016

Supplementary Reading: For basic Dirac notation quantum mechanics:

- Section 2.2 of M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*
- Sections 1.1–1.16 of W.H. Louisell, *Quantum Statistical Properties of Radiation*.

Problem 2.1

Here we shall explore the use of wave plates to perform polarization transformations on a single photon. The polarization state of a $+z$-propagating, frequency-ω photon at $z = 0$ is characterized by a complex-valued unit vector,

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}, \tag{1}$$

such that $\mathrm{Re}[\mathbf{i}e^{-j\omega t}]$ describes the time evolution of the photon at $z = 0$ where

$$\mathbf{i}^\dagger \mathbf{i} = |\alpha_x|^2 + |\alpha_y|^2 = 1,$$

with

$$\mathbf{i}^\dagger \equiv \begin{bmatrix} \alpha_x^* & \alpha_y^* \end{bmatrix},$$

being the unit-length condition for $\mathbf{i}$.

(a) For our monochromatic photon, propagation through L m of material in which light of arbitrary polarization propagates at velocity c/n, where c is light speed in vacuum and n is the material's refractive index at frequency ω, leads to a phase delay $\phi = \omega nL/c$. Thus the time evolution of the photon at $z = L$ is given by $\mathrm{Re}[\mathbf{i}e^{-j\omega(t-nL/c)}] = \mathrm{Re}[\mathbf{i}'e^{-j\omega t}]$, where $\mathbf{i}' \equiv \mathbf{i}e^{j\phi}$.

Show that the polarization state $\mathbf{i}'$ is identical to the polarization state $\mathbf{i}$, i.e., the contour traced out by $\mathrm{Re}[\mathbf{i}e^{-j\omega t}]$ in the x-y plane is identical to that traced out by $\mathrm{Re}[\mathbf{i}'e^{-j\omega t}]$.

(b) Wave plates are made of birefringent materials, i.e., materials which have different velocities of propagation for light polarized along their principal axes. When these axes are aligned with x and y, respectively, propagation of a monochromatic photon—whose polarization at $z = 0$ is given by Eq. (1)—results in a new polarization at $z = L$,

$$\mathbf{i}' = \begin{bmatrix} \alpha_x e^{j\phi_x} \\ \alpha_y e^{j\phi_y} \end{bmatrix}, \tag{2}$$

where $\phi_x \equiv \omega n_x L/c$ and $\phi_y \equiv \omega n_y L/c$ give the respective phase shifts in terms of the propagation velocities c/n_x and c/n_y along the x and the y axes. A quarter-wave plate (QWP) is one for which $\phi_x - \phi_y = \pi/2$. Suppose that a photon of $+45°$ linear polarization,

$$\mathbf{i} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

is the input to a QWP whose principal axes are aligned with x and y, respectively.

Show that the output of this QWP is circularly polarized.

Suppose that this circularly polarized output is the input to *another* QWP whose principal axes are aligned with x and y, respectively. What is the resulting polarization of the output from this QWP?

(c) A half-wave plate (HWP) is one for which the phase difference between propagation along its principal axes is π rad. Suppose that a photon of polarization

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is the input to an HWP whose "fast" (low refractive index) axis is parallel to the unit vector

$$\vec{i}_{\text{fast}} = \vec{i}_x \cos(\theta) + \vec{i}_y \sin(\theta),$$

and whose "slow" (high refractive index) axis is parallel to the unit vector

$$\vec{i}_{\text{slow}} = -\vec{i}_x \sin(\theta) + \vec{i}_y \cos(\theta).$$

What is the polarization state at the output of the HWP?

(d) Suppose we wish to transform an x-polarized input photon,

$$\mathbf{i}_{\text{in}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into an output photon of polarization state,

$$\mathbf{i}_{\text{out}} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}$$

Show that this can be done by first using a half-wave plate to transform $\mathbf{i}_{\text{in}}$ to

$$\mathbf{i}_{\text{HWP}} = \begin{bmatrix} |\alpha_x| \\ |\alpha_y| \end{bmatrix},$$

and then using another wave plate, whose principal axes are aligned with x and y respectively, and whose propagation phase difference $\phi_x - \phi_y$ is chosen appropriately, to transform $\mathbf{i}_{\text{HWP}}$ into $\mathbf{i}_{\text{out}}$.

(e) The polarization transformation scheme you verified in (d) is not a convenient experimental approach, because it requires a phase plate with a controllable propagation phase difference $\phi_x - \phi_y$. Here we consider an alternative approach that only needs a QWP and an HWP. Suppose that we wish to transform an arbitrary given input polarization

$$\mathbf{i}_{\text{in}} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

which is *not* linear, into horizontal polarization

$$\mathbf{i}_{\text{out}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Because $\mathbf{i}_{\text{in}}$ is, in general, an elliptical polarization, there must be a Cartesian coordinate system, (x', y'), in which this input polarization takes the form

$$\mathbf{i}_{\text{in}} = \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix},$$

with $\alpha'_y = jk\alpha'_x$, for k a positive constant. Use this fact to argue that a QWP, with its fast axis aligned in the y' direction, will convert $\mathbf{i}_{\text{in}}$ into linear polarization, after which an HWP can be used to obtain an $\mathbf{i}_{\text{out}}$ that is linearly polarized in the x direction. Using these results, explain how propagation through an HWP and a QWP can be used to transform an initially x-polarized photon into any desired polarization state.

Problem 2.2

Here we shall study the Poincaré sphere, viz., a 3-D real representation for the 2-D polarization state

$$\mathbf{i} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

of a $+z$-propagating, frequency-ω photon. Define a real-valued 3-vector, $\mathbf{r}$ as follows,

$$\mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\alpha_x^*\alpha_y] \\ 2\text{Im}[\alpha_x^*\alpha_y] \\ |\alpha_x|^2 - |\alpha_y|^2 \end{bmatrix}.$$

(a) Show that knowledge of $\mathbf{r}$ is equivalent to knowledge of $\mathbf{i}$, i.e., $\mathbf{r}$ completely describes photon's polarization.

(b) Show that $\mathbf{i}^\dagger\mathbf{i} = 1$ implies that $\mathbf{r}^T\mathbf{r} \equiv r_1^2 + r_2^2 + r_3^2 = 1$, i.e., the photon's polarization-state lies on the unit-sphere (called the Poincaré sphere) in $\mathbf{r}$ space.

(c) Where do x and y polarizations appear on the Poincaré sphere? Where do left and right circular polarizations appear on this sphere?

(d) Let

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \quad \text{and} \quad \mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\mathrm{Re}[\alpha_x^* \alpha_y] \\ 2\mathrm{Im}[\alpha_x^* \alpha_y] \\ |\alpha_x|^2 - |\alpha_y|^2 \end{bmatrix}$$

be equivalent representations of the polarization state of a monochromatic photon, and let

$$\mathbf{i}' \equiv \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix} \quad \text{and} \quad \mathbf{r}' \equiv \begin{bmatrix} r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \begin{bmatrix} 2\mathrm{Re}[\alpha'^*_x \alpha'_y] \\ 2\mathrm{Im}[\alpha'^*_x \alpha'_y] \\ |\alpha'_x|^2 - |\alpha'_y|^2 \end{bmatrix}$$

be another pair of equivalent polarizations. Show that

$$|\mathbf{i}'^\dagger \mathbf{i}|^2 = \frac{1 + \mathbf{r}'^T \mathbf{r}}{2}.$$

Problem 2.3

Let $\hat{A}$ be a linear operator that maps kets in the Hilbert space $\mathcal{H}$ into other kets in this space, i.e., for every $|x\rangle \in \mathcal{H}$, there is a $|y\rangle \in \mathcal{H}$ that satisfies $|y\rangle = \hat{A}|x\rangle$. Let $\{\, |\phi_n\rangle : n = 1, 2, \ldots, \}$ be an arbitrary complete orthonormal (CON) set of kets in $\mathcal{H}$, i.e.,

$$\begin{aligned} \langle \phi_n | \phi_m \rangle &= \delta_{nm} \equiv \begin{cases} 1, & \text{for } n = m, \\ 0, & \text{for } n \neq m. \end{cases} \\ \hat{I} &= \sum_{n=1}^{\infty} |\phi_n\rangle\langle\phi_n|, \end{aligned}$$

where $\hat{I}$ is the identity operator on $\mathcal{H}$.

(a) Show that the operator $\hat{A}$ is completely characterized by its $\{\phi_n\}$ matrix elements, viz., $\{\, \langle \phi_m | \hat{A} | \phi_n \rangle : 1 \leq n, m \leq \infty \,\}$, by proving that

$$\hat{A} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle \phi_m | \hat{A} | \phi_n \rangle |\phi_m\rangle\langle\phi_n|$$

(b) Let $|x\rangle = \sum_{n=1}^{\infty} x_n |\phi_n\rangle$ be an arbitrary ket in $\mathcal{H}$ and let $|y\rangle = \hat{A}|x\rangle$. Show that

$$|y\rangle = \sum_{m=1}^{\infty} y_m |\phi_m\rangle \quad \text{with} \quad y_m = \sum_{n=1}^{\infty} \langle \phi_m | \hat{A} | \phi_n \rangle x_n, \quad \text{for } 1 \leq n, m < \infty.$$

(c) Specialize your results from (a) and (b) to the case in which $\hat{A}$ is an observable, and the $\{\phi_n\}$ are its CON eigenkets.

Problem 2.4
Consider a quantum system, $\mathcal{S}$, in the Schrödinger picture, with Hamiltonian $\hat{H}$. Suppose that $\hat{H}$ has distinct, real-valued, non-negative, discrete eigenvalues $\{\, h_n : n = 0, 1, 2, \ldots, \}$ and associated orthonormal eigenkets, $\{\, |h_n\rangle : n = 0, 1, 2, \ldots, \}$.

(a) Show that the time-evolution operator obeys

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \exp[-jh_n(t - t_0)/\hbar] |h_n\rangle\langle h_n|, \quad \text{for } t \geq t_0.$$

(b) Show that

$$\left[\hat{U}(t, t_0), \hat{H}\right] = \left[\hat{U}^\dagger(t, t_0), \hat{H}\right] = 0,$$

i.e., the time-evolution operator and its adjoint both commute with the Hamiltonian.

(c) Suppose that the system is in the state $|\psi(t_0)\rangle = |h_1\rangle$ at time $t = t_0$. Find the state of the system $|\psi(t)\rangle$ at an arbitrary later time t.

(d) Suppose that $|\psi(t)\rangle$ is as found in (c), and that we measure the observable

$$\hat{O} = \sum_{k=1}^{\infty} o_k |o_k\rangle\langle o_k|$$

at time t. Find $\Pr(\hat{O}\text{-measurement outcome} = o_k)$ for $k = 1, 2, 3, \ldots$ Use this result to explain why the eigenkets of $\hat{H}$ are called stationary states.

Problem 2.5
Here we shall derive the time-frequency uncertainty principle of classical signal analysis. Essentially the same derivation can lead to the Heisenberg uncertainty principle for position and momentum by means of wave function (rather than Dirac-notation) quantum mechanics. Let $x(t)$ be a complex-valued, square-integrable time function whose Fourier transform is

$$X(f) \equiv \int_{-\infty}^{\infty} dt\, x(t) e^{-j2\pi f t}.$$

Define a normalized intensity for $x(t)$ via,

$$p(t) \equiv \frac{|x(t)|^2}{\displaystyle\int_{-\infty}^{\infty} dt\, |x(t)|^2},$$

and a normalized intensity for $X(f)$ via,

$$P(f) \equiv \frac{|X(f)|^2}{\displaystyle\int_{-\infty}^{\infty} df\, |X(f)|^2}.$$

(a) Show that $p(t)$ and $P(f)$ can be thought of as probability density functions, i.e., they are non-negative functions that integrate to one.

(b) Define the root-mean-square time duration for $x(t)$ to be,

$$T \equiv \sqrt{\int_{-\infty}^{\infty} dt\, t^2 p(t)},$$

and the root-mean-square bandwidth of $X(f)$ to be,

$$W \equiv \sqrt{\int_{-\infty}^{\infty} df\, f^2 P(f)}.$$

Show that

$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} df\, j2\pi f X(f) e^{j2\pi f t},$$

i.e., $j2\pi f X(f)$ is the Fourier transform of $\frac{dx(t)}{dt}$. Then, use Parseval's theorem and the Schwarz inequality and to prove that

$$TW \geq \frac{1}{2\pi} \frac{\left| \displaystyle\int_{-\infty}^{\infty} dt\, t x^*(t) \frac{dx(t)}{dt} \right|}{\displaystyle\int_{-\infty}^{\infty} dt\, |x(t)|^2}.$$

(c) Use the result from (b) and the fact that $|z| \geq |\mathrm{Re}(z)|$, for any complex number z, to show that,

$$\begin{aligned} TW &\geq \frac{1}{2\pi} \frac{\left| \mathrm{Re}\left(\displaystyle\int_{-\infty}^{\infty} dt\, t x^*(t) \frac{dx(t)}{dt} \right) \right|}{\displaystyle\int_{-\infty}^{\infty} dt\, |x(t)|^2} \\ &= \frac{1}{4\pi} \frac{\left| \displaystyle\int_{-\infty}^{\infty} dt\, t \frac{d(|x(t)|^2)}{dt} \right|}{\displaystyle\int_{-\infty}^{\infty} dt\, |x(t)|^2} = \frac{1}{4\pi}. \end{aligned}$$

(d) Show that equality occurs in (b) if and only if $x(t) = K\exp(at^2)$, where K and a are complex-valued constants with $\mathrm{Re}(a) < 0$. Assume that $x(t)$ is of this form and then show that equality occurs in (c) if and only if a is real. Verify that

$$x(t) = \frac{\exp(-t^2/4t_0^2)}{(2\pi t_0^2)^{1/4}},$$

has Fourier transform

$$X(f) = (8\pi t_0^2)^{1/4}\exp(-4\pi^2 f^2 t_0^2),$$

and that this $x(t)$ has $T = t_0$ and this $X(f)$ has $W = 1/4\pi t_0$, thus giving $TW = 1/4\pi$.