LECTURE 20

Last time:

- Gaussian channels with feedback
- Upper bound to benefit of capacity

Lecture outline

- Multiple access channels
- Coding theorem
- Capacity region for Gaussian channels

Reading: Section 14.1-14.3.

Multiple access channels

Several users share the same medium

What is the right metric? Joint information?

User 1 has rate R_1 and user 2 has rate R_2

How do we relate them to mutual information?

Model: $Y_i = X_{1i} + X_{2i} + N_i$

Liao and Ahlswede (independently, 1972)

$R_1 \leq I(X_1; Y | X_2)$

$R_2 \leq I(X_2; Y | X_1)$

$R_1 + R_2 \leq I((X_1, X_2); Y)$

Coding theorem

m_i: message sent by user i

$\widehat{m}_i$: decoded message for user i

Pe_1 (Pe_2): probability that the decoded codeword for user 1 (2) is different from that sent by user 1 (2) while the decoded codeword for user 2 (1) is the same as the one sent by user 2 (1) (such errors will be denoted as errors of type 1 (2))

$Pe_{1,2}$: probability that the decoded codewords for both users 1 and 2 are different from those sent by those users (such an error will be denoted as error of type 3)

We begin by bounding the probability of error with an exponential argument and then we explore the behavior of that argument

Coding theorem

We first consider errors of type 1 - results for errors of type 2 can be derived analogously

We denote Pe_{1,m_1,m_2} the probability that an error of type 1 occurs conditioned on messages m_1 and m_2 being sent

Using the overbar to denote expectation

$$\overline{Pe_{1,m_1,m_2}}$$
$$= \int_{\underline{y}} \int_{\underline{x}_1} \int_{\underline{x}_2} f_{\underline{X}_1}(\underline{x}_1) f_{\underline{X}_2}(\underline{x}_2) f_{\underline{Y}|\underline{X}_1,\underline{X}_2}(y|x_1,x_2)$$
$$P\left(\left\{(\widehat{m}_1 \neq m_1) \cap \left(\widehat{m}_2 = m_2|\underline{y},\underline{x}_1,\underline{x}_2\right)\right\}\right) d\underline{x}_2 d\underline{x}_1 d\underline{y}.$$

Using the union bound, we obtain

$$P\left(\left\{(\widehat{m}_1 \neq m_1) \cap \left(\widehat{m}_2 = m_2|\underline{y},\underline{x}_1,\underline{x}_2\right)\right\}\right) \leq$$
$$\left\{\sum_{m \neq m_1} P\left(\left\{(\widehat{m}_1 = m) \cap \left(\widehat{m}_2 = m_2|\underline{y},\underline{x}_1,\underline{x}_2\right)\right\}\right)\right\}^{\rho}$$

$\forall 0 \leq \rho \leq 1$.

Coding theorem

Using arguments similar to those for the single user strong coding theorem, we can establish

$\forall \rho \in [0,1]$, $f_{\underline{X}_1}(\underline{x}_1)$ and $f_{\underline{X}_2}(\underline{x}_2)$ probability density functions for $\underline{X_1}$ and $\underline{X_2}$, respectively, we have

$$\overline{Pe_{1,m_1,m_2}} \leq$$
$$exp\left(-N\left(-\rho R_1 + E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right)\right)\right)$$

where we have defined,

$$E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right) = -\frac{1}{N}$$
$$\ln\left\{\int_{\underline{y}}\int_{\underline{x}_2} f_{\underline{X}_2}(\underline{x}_2)\right.$$
$$\left.\left\{\int_{\underline{x}} f_{\underline{X}_1}(\underline{x})\, f_{\underline{Y}|\underline{X},\underline{X}_2}\left(\underline{y}|\underline{x},\underline{x}_2\right)^{\frac{1}{1+\rho}} d\underline{x}\right\}^{1+\rho} d\underline{x}_2 d\underline{y}\right\}$$

Coding theorem

It now suffices to determine the behavior of the exponent to determine whether the upper bound to error probability becomes vanishingly small

The following lemma parallels the one for the one-user case

If $I(\underline{X}_1;\underline{Y}|\underline{X}_2) > 0$, then for all $1 \geq \rho \geq 0$ we have

$$I(\underline{X}_1;\underline{Y}|\underline{X}_2) \geq \frac{\partial N E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right)}{\partial \rho} > 0 \quad (1)$$

$$E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right) \geq 0 \quad (2)$$

$$\frac{\partial^2 N E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right)}{\partial \rho^2} \leq 0 \quad (3)$$

$$\left.\frac{\partial E_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right)}{\partial \rho}\right|_{\rho=0} = \frac{I(\underline{X}_1;\underline{Y}|\underline{X}_2)}{N}. \quad (4)$$

Coding theorem

Let $\mathbf{q}^{1,N}$, $\mathbf{q}^{2,N}$ be a pair of probability density functions for the codewords of length N of users 1 and 2

In order for $E_0^1\left(\rho, \mathbf{q}^{1,N}, \mathbf{q}^{2,N}\right) - \rho R_1$ to be strictly positive for some ρ in $[0,1]$, it is necessary and sufficient that

$$\left\{\frac{\partial E_0^1\left(\rho, \mathbf{q}^{1,N}, \mathbf{q}^{2,N}\right)}{\partial \rho} - R_1\right\}\Bigg|_{\rho=0} > 0.$$

We have that:

For all $f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)$ probability density functions for $\underline{X_1}$, $\underline{X_2}$, we have

$$\frac{I(\underline{X}_1;\underline{Y}|\underline{X}_2)}{N} > R_1 \geq 0$$

$$\Rightarrow \exists \rho \in [0,1] \text{ s.t.}$$

$$\mathsf{E}_0^1\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right) - R_1\rho > 0$$

We can establish analogous results for errors of type 2 and 3

Coding theorem

Let us define

$$E_{min} = \quad min\left[max_\rho\left(E_0^1\left(\rho, q^{1,N}, q^{2,N}\right) - R_1\rho\right),\right.$$
$$max_\rho\left(E_0^2\left(\rho, f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)\right) - R_2\rho\right),$$
$$\left. max_\rho\left(E_0^3\left(\rho, q^{1,N}, q^{2,N}\right) - (R_1 + R_2)\rho\right)\right]$$

where E_0^2 and E_0^3 is defined analogously to E_0^1. We may state the following theorem:

For all$f_{\underline{X}_1}(\underline{x}_1), f_{\underline{X}_2}(\underline{x}_2)$ probability density functions for $\underline{X_1}$, $\underline{X_1}$, for any messages m_1 and m_2 of users 1 and 2, we have

$$Pe_{m_1,m_2} \leq 3e^{-NE_{min}}$$

and

$$\frac{I\left(\underline{X_1}; \underline{Y} | \underline{X_2}\right)}{N} > R_1 \geq 0 \text{ and}$$
$$\frac{I\left(\underline{X_2}; \underline{Y} | \underline{X_1}\right)}{N} > R_2 \geq 0 \text{ and}$$
$$\frac{I\left(\left(\underline{X_1}, \underline{X_2}\right); \underline{Y}\right)}{N} > R_1 + R_2 \geq 0 \Rightarrow E_{min} > 0$$

Capacity region

Cover-Wyner region for two users

Capacity region

Consider AWGN Multiple-access channel, user i has energy $\sigma^2_{X_1}$

Pentagon: dominant face corresponds to

$$\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_1}+\sigma^2_{X_2}}{\sigma^2_N}\right)$$

Interference cancellation at the corners:

$$\left(\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_1}}{\sigma^2_{X_2}+\sigma^2_N}\right),\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_2}}{\sigma^2_N}\right)\right)$$

$$\left(\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_1}}{\sigma^2_N}\right),\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_2}}{\sigma^2_{X_1}+\sigma^2_N}\right)\right)$$

without interference cancellation:

$$\left(\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_1}}{\sigma^2_{X_2}+\sigma^2_N}\right),\frac{1}{2}\ln\left(1+\frac{\sigma^2_{X_2}}{\sigma^2_{X_1}+\sigma^2_N}\right)\right)$$

Recall DS-CDMA example

Capacity region

FDMA: $\left(\frac{W_1}{2}\ln\left(1+\frac{\sigma_{X_1}^2}{W_1\sigma_N^2}\right), \frac{W_2}{2}\ln\left(1+\frac{\sigma_{X_2}^2}{W_2\sigma_N^2}\right)\right)$

for equal energies, equal Ws desirable

TDMA: let α be the fraction of time that user 1 transmits

$$\left(\frac{\alpha}{2}\ln\left(1+\frac{\sigma_{X_1}^2}{\alpha\sigma_N^2}\right), \frac{1-\alpha}{2}\ln\left(1+\frac{\sigma_{X_2}^2}{(1-\alpha)\sigma_N^2}\right)\right)$$

for equal energies, $\alpha = 0.5$ desirable

How do we achieve points on the dominant face, that yields maximum sum rate?

First way: time share between the corners

Other way: rate splitting

Capacity region

Make one user (say user 1) into two virtual users (virtual user 1 and virtual user 3) and split energy between these two virtual users

Virtual user 1 rate:

$$\frac{1}{2}\ln\left(1+\frac{\alpha\sigma_{X_1}^2}{\sigma_N^2+(1-\alpha)\sigma_{X_1}^2+\sigma_{X_2}^2}\right)$$

User 2 rate:

$$\frac{1}{2}\ln\left(1+\frac{\sigma_{X_2}^2}{\sigma_N^2+(1-\alpha)\sigma_{X_1}^2}\right)$$

Virtual user 3 rate:

$$\frac{1}{2}\ln\left(1+\frac{(1-\alpha)\sigma_{X_1}^2}{\sigma_N^2}\right)$$

Capacity region

We have

$$
\begin{aligned}
& \frac{1}{2}\ln\left(1+\frac{\alpha\sigma_{X_1}^2}{\sigma_N^2+(1-\alpha)\sigma_{X_1}^2+\sigma_{X_2}^2}\right) \\
+ \; & \frac{1}{2}\ln\left(1+\frac{\sigma_{X_2}^2}{\sigma_N^2+(1-\alpha)\sigma_{X_1}^2}\right) \\
+ \; & \frac{1}{2}\ln\left(1+\frac{(1-\alpha)\sigma_{X_1}^2}{\sigma_N^2}\right) \\
= \; & \frac{1}{2}\ln\left(1+\frac{\sigma_{X_1}^2}{\sigma_{X_2}^2+\sigma_N^2}\right)+\frac{1}{2}\ln\left(1+\frac{\sigma_{X_2}^2}{\sigma_N^2}\right) \\
= \; & \frac{1}{2}\ln\left(1+\frac{\sigma_{X_1}^2+\sigma_{X_2}^2}{\sigma_N^2}\right)
\end{aligned}
$$

Capacity region

If we have

$$R_2 = \frac{1}{2}\ln\left(1 + \frac{\sigma_{X_2}^2}{\sigma_N^2 + (1-\alpha)\sigma_{X_1}^2}\right),$$

then R_1 is defined as

$$\begin{aligned} & \frac{1}{2}\ln\left(1 + \frac{\sigma_{X_1}^2 + \sigma_{X_2}^2}{\sigma_N^2}\right) - R_2 \\ = \ & \frac{1}{2}\ln\left(1 + \frac{\alpha\sigma_{X_1}^2}{\sigma_N^2 + (1-\alpha)\sigma_{X_1}^2 + \sigma_{X_2}^2}\right) \\ + \ & \frac{1}{2}\ln\left(1 + \frac{(1-\alpha)\sigma_{X_1}^2}{\sigma_N^2}\right) \end{aligned}$$

One variable provides all the necessary degrees of freedom

Capacity region

In general, for μ users, the capacity region is

$$\sum_{i\in\mathcal{S}} R_i \leq I((X_i)_{i\in\mathcal{S}}; Y | (X_i)_{i\notin\mathcal{S}}), \forall\mathcal{S} \subset \{1, \ldots, \mu\}$$

We have $2\mu - 1$ pseudo-users are sufficient to achieve any point on the multiple-access dominant face