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Recitation 1

BACKGROUND MATERIAL ON SETS AND REAL ANALYSIS

1 SETS

A **set** is a collection of objects, which are the **elements** of the set. If A is a set and x is an element of A, we write $x \in A$. If x is not an element of A, we write $x \notin A$. A set can have no elements, in which case it is called the **empty set**, denoted by Ø.

Sets can be specified in a variety of ways. If A contains a finite number of elements, say $x_1, x_2, \ldots, x_n$, we write it as a list of the elements, in braces:

$$A = \{x_1, x_2, \ldots, x_n\}.$$

For example, the set of possible outcomes of a die roll is $\{1, 2, 3, 4, 5, 6\}$, and the set of possible outcomes of a coin toss is $\{H, T\}$, where H stands for "heads" and T stands for "tails."

More generally, we can consider the set of all x that have a certain property P, and denote it by

$$\{x \mid x \text{ satisfies } P\}.$$

(The symbol "$\mid$" is to be read as "such that.") For example, the set of even integers can be written as $\{k \mid k/2 \text{ is integral}\}$. Similarly, the set of all real numbers x in the interval $[0, 1]$ can be written as $\{x \mid 0 \leq x \leq 1\}$.

If A contains infinitely many elements $x_1, x_2, \ldots$, that can be enumerated in a list (so that the elements are in a one-to-one correspondence with the positive integers), we write

$$A = \{x_1, x_2, \ldots\},$$

and we say that A is **countably infinite**. For example, the set of even integers can be written as $\{0, 2, -2, 4, -4, \ldots\}$, and is countably infinite. The term **countable** is sometimes used to refer to a set which is either finite or countably infinite. A set which is not countable is said to be **uncountable**.

If every element of a set A is also an element of a set B, we say that A is a **subset** of B, and we write $A \subset B$ or $B \supset A$. If $A \subset B$ and $A \subset B$, the two sets are **equal**, and we write $A = B$.[1] It is sometimes expedient to introduce a **universal set**, denoted by Ω, which contains all objects that could conceivably be of interest in a particular context. Having specified a context in terms of a universal set Ω, one then only considers sets A that are subsets of Ω.

2 SET OPERATIONS

The **complement** of a set A, with respect to a universal set Ω, is the set $\{x \in \Omega \mid x \notin A\}$ of all elements of Ω that do not belong to A, and is denoted by A^c. Note that $\Omega^c = \text{Ø}$.

The **union** of two sets A and B is the set of all elements that belong to A or B (or both), and is denoted by $A \cup B$. The **intersection** of two sets A and B is the set of all elements that belong to both A and B, and is denoted by $A \cap B$. Thus,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

and

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

We also define

$$A \setminus B = A \cap B^c = \{x \mid x \in A \text{ and } x \notin B\},$$

which is the set of all elements that belong to A but not in B.

We will often deal with the union or the intersection of several, even infinitely many sets, defined in the obvious way. In particular, if I is a (possibly infinite) index set, and for each $i \in I$ we have a set A_i, the union of these sets is defined as

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\},$$

and their intersection is defined as

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$

In case we are dealing with the union or intersection of countably many sets A_i, the notation $\cup_{i=1}^{\infty} A_i$ and $\cup_{i=1}^{\infty} A_i$, respectively, is used.

Two sets are said to be disjoint if their intersection is empty. More generally, several sets are said to be **disjoint** if no two of them have a common element.

[1] Some texts use the notation $A \subseteq B$ to indicate that A is a subset of B, and reserve the notation $A \subset B$ for the case where A is a **proper** subset of B, i.e., a subset of B which is not equal to B.

Disjoint sets are also said to be **mutually exclusive**. A collection of sets is said to be a **partition** of a set A if the sets in the collection are disjoint and their union is A.

2.1 The Algebra of Sets

Set operations have several properties, which are elementary consequences of the definitions. Some examples are:

$$\begin{aligned} A \cup B &= B \cup A, & A \cap B &= B \cap A, \\ A \cup (B \cup C) &= (A \cup B) \cup C, & A \cap (B \cap C) &= (A \cap B) \cap C, \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), & A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ (A^c)^c &= A, & A \cap A^c &= \emptyset, \\ A \cup \Omega &= \Omega, & A \cap \Omega &= A. \end{aligned}$$

Two particularly useful properties are given by **De Morgan's laws** which state that

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c, \qquad \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

To establish the first law, suppose that $x \in (\cup_{i \in I} A_i)^c$. Then, $x \notin \cup_{i \in I} A_i$, which implies that for every $i \in I$, we have $x \notin A_i$. Thus, x belongs to the complement of every A_i, and $x \in \cap_{i \in I} A_i^c$. This shows that $(\cup_{i \in I} A_i)^c \subset \cap_{i \in I} A_i^c$. The reverse inclusion is established by reversing the above argument, and the first law follows. The argument for the second law is similar.

3 NOTATION: SOME COMMON SETS

We now introduce the notation that will be used to refer to some common sets:

(a) $\mathbb{R}$ denotes the set of all **real numbers**;

(b) $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{-\infty, \infty\}$, the set of **extended real numbers**.

(c) $\mathbb{Z}$ denotes the set of all **integers**;

(d) $\mathbb{N}$ denotes the set of **natural numbers** (the positive integers).

Also, for any $a, b \in \overline{\mathbb{R}}$, we use the following notation:

(a) $[a, b]$ denotes the set $\{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$;

(b) (a, b) denotes the set $\{x \in \overline{\mathbb{R}} \mid a < x < b\}$;

(c) $[a, b)$ denotes the set $\{x \in \overline{\mathbb{R}} \mid a \leq x < b\}$;

(d) $(a, b]$ denotes the set $\{x \in \overline{\mathbb{R}} \mid a < x \leq b\}$.

We finally introduce some definitions related to products of sets.

(a) The **Cartesian product** of n sets $A_1, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, or $\prod_{i=1}^{n} A_i$ for short, is the set of all n-tuples that can be formed by picking one element from each set, that is,

$$\prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid a_i \in A_i, \ \forall\, i\}.$$

The set $A \times A$ is also denoted by A^2. The notation A^n is defined similarly.

(b) The Cartesian product $\prod_{i=1}^{\infty} A_i$ of an infinite sequence of sets A_i is defined as the set of all sequences $(a_1, a_2, \ldots)$ where $a_i \in A_i$ for each i. The simpler notation A^∞ is used if $A_i = A$ for all i.

(c) The set of all subsets of a set A is denoted by 2^A.

(d) Given two sets A and B, A^B stands for the set of functions from B to A.

As defined above, a sequence $(a_1, a_2, \ldots)$ of elements of a set A belongs to A^∞. However, such a sequence can also be viewed as a function from $\mathbb{N}$ into A, which belongs to $A^{\mathbb{N}}$. Thus, there is a one-to-one correspondence between A^∞ and $A^{\mathbb{N}}$.

In the special case where $A = \{0, 1\}$, a sequence $(a_1, a_2, \ldots)$ can be identified with a subset of $\mathbb{N}$, namely the set $\{n \in \mathbb{N} \mid a_n = 1\}$. We conclude that there is a one-to-one correspondence between $\{0, 1\}^\infty$, $\{0, 1\}^{\mathbb{N}}$, and $2^{\mathbb{N}}$.

4 REMARKS ON THE CARDINALITY OF SETS

For any finite set, its cardinality is intuitively defined as the number of elements in it. For infinite sets, one might be tempted to define their cardinality as infinity. However, not all infinite sets are created equal, so we need to find a way to differentiate them. We begin by defining a way to compare the cardinality of any two arbitrary sets A and B.

Definition 1.

(a) We say that A and B have the same cardinality, denoted by $|A| = |B|$, if and only if there exists a bijective function $f : A \to B$.

(b) We say that A has cardinality smaller than or equal to B, denoted by $|A| \leq |B|$, if and only if there exists an injective function $g : A \to B$.

(c) We say that A has cardinality bigger than or equal to B, denoted by $|A| \geq |B|$, if and only if there exists an surjective function $g : A \to B$.

Using this definition, we define what it means for a set to be countable, or uncountable.

Definition 2.

(a) We say that a set is countable, if its cardinality is smaller than or equal to the set of natural numbers $\mathbb{N}$.

(b) We say that a set is uncountable, if its cardinality is strictly bigger than the cardinality of the natural numbers $\mathbb{N}$.

Since the *continuum hypothesis* (a set theoretic axiom) states that there is no set with cardinality between the cardinality of the natural numbers and the cardinality of the real numbers, uncountable sets can also be defined as the ones that have cardinality bigger than or equal to the cardinality of the real numbers.

We collect here some facts that are useful in distinguishing countable and uncountable sets.

Theorem 1.

(a) *The union of countably many countable sets is a countable set.*

(b) *If A is finite, of cardinality n, then 2^A has cardinality 2^n.*

(c) *The Cartesian product of finitely many countable sets is countable.*

(d) *The set of rational numbers is countable.*

(e) *The set $\{0, 1\}^\infty$ is uncountable.*

(f) *The Cartesian product of infinitely many sets (with at least two elements each) is uncountable.*

Proof.

(a) Left as an exercise.

(b) When choosing a subset of A, there are two choices for each element of A: whether to include it in the subset or not. Since there are n elements, with two choices for each, the total number of choices is 2^n.

(c) Suppose that A and B are countable sets, and that $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$. We observe that

$$A \times B = \bigcup_{i=1}^{\infty} \big(\{a_i\} \times B\big).$$

For any i, there is a one-to-one correspondence between elements of B and elements of $\{a_i\} \times B$. Therefore $\{a_i\} \times B$ is countable. Using part (a) of the theorem, it follows that $A \times B$ is countable.

We continue by induction. We fix some $k \geq 2$ and use the induction hypothesis that the Cartesian product of k or fewer countable sets is countable. Suppose that the sets $A_1, \ldots, A_{k+1}$ are countable. We observe that the set $A_1 \times \cdots \times A_{k+1}$ is essentially the same as the set $(A_1 \times \cdots \times A_k) \times A_{k+1}$, which is a Cartesian product of two sets. The first is countable, by the induction hypothesis; the second is countable by assumption. The result follows.

(d) Left as an exercise.

(e) Suppose, in order to derive a contradiction, that the elements of $\{0, 1\}^\infty$ (each of which is a binary sequences) can be arranged in a sequence $s_1, s_2, \ldots$. Consider the binary sequence s whose kth entry is chosen to be different from the kth entry of the sequence s_k. This sequence s is certainly an element of $\{0,1\}^\infty$, but is different from each of the sequences s_k, by construction. This means that the sequence $s_1, s_2, \ldots$ cannot exhaust all of the elements of $\{0,1\}^\infty$ and therefore the latter set is uncountable.

(f) Follows from (e) since 2^A has at least as many elements as $2^{\mathbb{N}}$, which can be identified with $\{0,1\}^\infty$. $\square$

5 SEQUENCES AND LIMITS

Formally, a sequence of elements of a set A is a mapping $f : \mathbb{N} \to A$. Let $a_i = f(i)$. The corresponding sequence is often written as $(a_1, a_2, \ldots)$ or $\{a_k\}$ for short.

Given a sequence $\{a_k\}$ and an increasing sequence of natural numbers $\{k_i\}$, we can construct a new sequence whose ith element is a_{k_i}. This new sequence is called a **subsequence** of $\{a_k\}$. Informally, a subsequence of $\{a_k\}$ is obtained by skipping some of the elements of the original sequence.

Definition 3.

(a) *A sequence* $\{x_k\}$ *of real numbers (also called a "real sequence") is said to* **converge** *to a real number* x *if for every* $\epsilon > 0$ *there exists some (positive integer)* K *such that* $|x_k - x| < \epsilon$ *for every* $k \geq K$.

(b) *A real sequence* $\{x_k\}$ *is said to converge to* ∞ *(respectively,* $-\infty$*) if for every real number* c *there exists some* K *such that* $x_k \geq c$ *(respectively,* $x_k \leq c$*) for all* $k \geq K$.

(c) *If a real sequence converges to some* x *(possibly infinite), we say that* x *is the* **limit** *of* x_k*; symbolically,* $\lim_{k\to\infty} x_k = x$.

(d) *A real sequence* $\{x_k\}$ *is called a* **Cauchy sequence** *if for every* $\epsilon > 0$*, there exists some* K *such that* $|x_k - x_m| < \epsilon$ *for all* $k \geq K$ *and* $m \geq K$.

(e) *A real sequence* $\{x_k\}$ *is said to be* **bounded above** *(respectively,* **below***) if there exists some real number* c *such that* $x_k \leq c$ *(respectively,* $x_k \geq c$*) for all* k.

(f) *A real sequence* $\{x_k\}$ *is called* **bounded** *if the sequence* $\{|x_k|\}$ *is bounded above.*

(g) *A real sequence is said to be* **nonincreasing** *(respectively,* **nondecreasing***) if* $x_{k+1} \leq x_k$ *(respectively,* $x_{k+1} \geq x_k$*) for all* k*. A sequence that is either nonincreasing or nondecreasing is called* **monotonic**.

The following result is a fundamental property of the real-number system, and is presented without proof.

Theorem 2. *Every monotonic real sequence converges to an extended real number. If the sequence is also bounded, then it converges to a real number.*

We continue with the definition of some key quantities associated with sets or sequences of real numbers.

Definition 4.

(a) *The* **supremum** *(or* **least upper bound***) of a set A of real numbers, denoted by* $\sup A$*, is defined as the smallest extended real number x such that $x \geq y$ for all $y \in A$.*

(b) *The* **infimum** *(or* **greatest lower bound***) of a set A of real numbers, denoted by* $\inf A$*, is defined as the largest extended real number x such that $x \leq y$ for all $y \in A$.*

(c) *Given a sequence $\{x_k\}$ of real numbers, the* **supremum** *of the sequence, denoted by* $\sup_k x_k$*, is defined as* $\sup\{x_k \mid k = 1, 2, \ldots\}$*. The* **infimum** *of a sequence is similarly defined.*

(d) *The* **upper limit** *of a real sequence $\{x_k\}$, denoted by* $\limsup_{k\to\infty} x_k$*, is defined to be equal to* $\lim_{m\to\infty} \sup\{x_k \mid k \geq m\}$*.*

(e) *The* **lower limit** *of a real sequence $\{x_k\}$, denoted by* $\liminf_{k\to\infty} x_k$*, is defined to be equal to* $\lim_{m\to\infty} \inf\{x_k \mid k \geq m\}$*.*

Remarks:

(a) It turns out that the supremum and infimum of a set of real numbers is guaranteed to exist. This is a direct consequence of the way the real-number system is constructed (see, e.g., [R]). It can also be proved by building on Theorem 2.

(b) The infimum or supremum of a set need not be an element of a set. For example, if $A = \{1/k \mid k \in \mathbb{N}\}$, then $\inf A = 0$, but $0 \notin A$.

(c) If $\sup A$ happens to also be an element of A, then $\sup A$ is the maximum (i.e., the largest element) of A, and in that case, it is also denoted as $\max A$. Similarly, if $\inf A$ is an element of A, it is the minimum of A, and is denoted as $\min A$.

(d) If a set or a sequence of real numbers has arbitrarily large elements (that is, no finite upper bound), then the supremum is equal to ∞. Similarly, if it has arbitrarily small elements (that is, no finite lower bound), then the infimum is equal to $-\infty$.

(e) A careful application of the definitions shows that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. On the other hand, if a set is nonempty, then $\inf A \leq \sup A$.

(f) A sequence need not have a limit (e.g., consider the sequence $x_n = (-1)^n$. On the other hand, the upper and lower limits of a real sequence are always defined. To see this, let $y_m = \sup\{x_k \mid k \geq m\}$. The sequence $\{y_m\}$ is nonicreasing and therefore has a (possibly infinite) limit. We have $\limsup_{m\to\infty} x_k = \lim_{m\to\infty} y_m$, and the latter limit is guaranteed to exist, by Theorem 2. A similar argument applies to the lower limit.

Theorem 3. *Let $\{x_k\}$ be a real sequence.*

(a) There holds

$$\inf_k x_k \leq \liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k \leq \sup_k x_k.$$

(b) The sequence $\{x_k\}$ converges (to an extended real number) if and only if $\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k$, and in that case, both of these quantities are equal to the limit of x_k.

The next definition refers to convergence of finite-dimensional real vectors.

Definition 5.

(a) A sequence $\{x_k\}$ of vectors in $\mathbb{R}^n$ is said to converge to some $x \in \mathbb{R}^n$ if the ith coordinate of x_k converges to the ith coordinate of x, for every i. The notation $\lim_{k\to\infty} x_k = x$ is used again.

(b) A sequence of vectors is called a **Cauchy sequence** *(respectively,* **bounded***) if each coordinate sequence is a Cauchy sequence (respectively, bounded).*

(c) We say that some $x \in \mathbb{R}^n$ is a **limit point** *of a sequence $\{x_k\}$ in $\mathbb{R}^n$ if there exists a subsequence of $\{x_k\}$ that converges to x.*

(d) Let A be a subset of $\mathbb{R}^n$. We say that $x \in \mathbb{R}^n$ is an **limit point** *of A if there exists a sequence $\{x_k\}$, consisting of elements of A, different from x, that converges to x.*

We summarize some key facts about convergence of vector-valued sequences, see, e.g., [R].

Theorem 4.

(a) A bounded sequence in $\mathbb{R}^n$ has at least one limit point.

(b) A bounded sequence in $\mathbb{R}^n$ converges if and only if it has a unique limit point (in which case, the limit point is also the limit of the sequence).

(c) A sequence in $\mathbb{R}^n$ converges to an element of $\mathbb{R}^n$ if and only if it is a Cauchy sequence.

(d) Let $\{x_k\}$ be a real sequence. If $\limsup_{k\to\infty} x_k$ (respectively, $\liminf_{k\to\infty} x_k$) is finite, then it is the largest (respectively, smallest) limit point of the sequence $\{x_k\}$.

6 LIMITS OF SETS

Consider a sequence $\{A_n\}$ of sets. There are several ways of defining what it means for the sequence to converge to some limiting set. The definitions that will be most useful for our purposes are given below.

Definition 6.

(a) We define $\limsup_{n\to\infty} A_n$ as the set of all elements ω that belong to infinitely many of the sets A_n. Formally,

$$\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \Big(\bigcup_{n=k}^{\infty} A_n \Big).$$

The notation $\{A_n$ i.o.$\} = \limsup_{n\to\infty} A_n$ is also used.

(b) We define $\liminf_{n\to\infty} A_n$ as the set of all ω that belong to all but finitely many of the sets A_n. Formally,

$$\liminf_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} \Big(\bigcap_{n=k}^{\infty} A_n \Big).$$

(c) We say that A is the **limit** *of the sequence A_n (symbolically, $A_n \to A$, or $\lim_{n\to\infty} A_n = A$) if $A = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$.*

Note that a sequence of sets A_n need not have a limit, but $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are always well defined.

In order to parse the above definitions, note that $\omega \in \cup_{n=k}^{\infty} A_n$ if and only if there exists some $n \geq k$ such that $\omega \in A_n$. We then see that ω belongs to the intersection $\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n$ if and only if for every k, there exists some $n \geq k$ such that $\omega \in A_n$; this is equivalent to requiring that ω belong to infinitely many of the sets A_n.

Similarly, $x \in \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} A_n$ if and only if for some k, x belongs to $\cap_{n=k}^{\infty} A_n$. Equivalently, for some k, x belongs to all of the sets $A_k, A_{k+1}, \ldots$, i.e., x belongs to all but finitely many of the sets A_n.

When, the sequence of sets $\{A_n\}$ is monotonic, the limits turn out to behave as expected.

Theorem 5.

(a) If A_n is an increasing sequence of sets ($A_n \subset A_{n+1}$, for all n), then $\lim_{n\to\infty} A_n$ exists and is equal to $\cup_{n=1}^{\infty} A_n$.

(b) If A_n is an decreasing sequence of sets ($A_n \supset A_{n+1}$, for all n), then $\lim_{n\to\infty} A_n$ exists and is equal to $\cap_{n=1}^{\infty} A_n$.

Reasoning about a sequence of functions is often easier than reasoning about the convergence of a sequence of sets. A link between the two notions of convergence is provided by the following.

Definition 7. *The indicator function $I_A : \Omega \to \{0, 1\}$ of a set A is defined by*

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

We then have the following result.

Theorem 6. *We have $\lim_{n\to\infty} A_n = A$ if and only $\lim_{n\to\infty} I_{A_n}(\omega) = I_A(\omega)$ for all ω.*

Proof. To prove one direction of the result, we assume that $\lim_{n\to\infty} A_n = A$. Consider the two following cases:

(i) Suppose that $\omega \in A$. Since $\liminf_{n\to\infty} A_n = A$, ω belongs to all but finitely many of the sets A_n, which implies that $I_{A_n}(\omega) = 1$ for all but finitely many n. This establishes that $\lim_{n\to\infty} I_{A_n}(\omega) = 1 = I_A(\omega)$.

(ii) Suppose now that $\omega \notin A$. Since $\limsup_{n\to\infty} A_n = A$, ω belongs to at most finitely many of the sets A_n, which implies that $I_{A_n}(\omega) = 0$ for all but finitely many n. This establishes that $\lim_{n\to\infty} I_{A_n}(\omega) = 0 = I_A(\omega)$, and one direction of the desired result has been proved.

To prove the reverse direction, we consider two cases.

(i) The limit $\lim_{n\to\infty} A_n$ exists, and is a set B different than A. Then either $B \setminus A$ or $A \setminus B$ is nonempty.

Suppose $B \setminus A$ is nonempty. Let $\omega \in B \setminus A$. Then, by the part of the result that has already been established, $\lim_{n\to\infty} I_{A_n}(\omega) = I_B(\omega) = 1$. However, $\omega \notin A$ and $A = \limsup A_n$, which means ω belongs to finitely many of the sets A_n. Thus, $\lim_{n\to\infty} I_{A_n}(\omega) = 0$, which is a contradiction.

Similarly, suppose $A \setminus B$ is nonempty. Let $\omega \in A \setminus B$. Then, by the part of the result that has already been established, $\lim_{n\to\infty} I_{A_n}(\omega) = I_B(\omega) = 0$. However, $\omega \in A$ and $A = \liminf A_n$, which means ω belongs to all but finitely many of the sets A_n. Thus, $\lim_{n\to\infty} I_{A_n}(\omega) = 1$, which is a contradiction.

(ii) The limit $\lim_{n\to\infty} A_n$ does not exist. In that case, we have $\liminf_{n\to\infty} A_n < \limsup_{n\to\infty} A_n$. This implies that there exists some ω that belongs to infinitely many of the sets A_n, but also does not belong to infinitely many of those sets. In that case, $I_{A_n}(\omega) = 0$, for infinitely many choices of n, and also $I_{A_n}(\omega) = 1$, for infinitely many choices of n. This implies that the sequence $I_{A_n}(\omega)$ does not converge, and therefore the condition $\lim_{n\to\infty} I_{A_n}(\omega) = I_A(\omega)$, for every ω, cannot hold.

$\square$

7 BOREL SETS

We define the Borel σ-algebra in $I = [0, 1]$ as the σ-algebra generated by the intervals of the form $[a, b] = \{x \in I \mid a \leq x \leq b\}$, where $0 \leq a \leq b \leq 1$.

Theorem 1.

(a) $\{x\}$ *is a Borel set for any* $x \in I$.

(b) The set of rational numbers in $[0, 1]$ *is a Borel set.*

(c) All sets of the form $(a, b] = \{x \in I \mid a < x \leq b\}$ *or* $(a, b) = \{x \in I \mid a < x < b\}$*, where* $0 \leq a < b \leq 1$*, are Borel sets.*

(d) If S *is an open set contained in* I*, then* S *is a Borel set.*

Proof. (a) This is because $\{x\}$ is a set of the form $[a, b]$ with $a = b = x$.

Another interesting approach could also be used: for all $x \in [0, 1)$,

$$\{x\} = \bigcap_{n=1}^{\infty} [x, x + (1/n)].$$

For $x = 1$, we write

$$\{1\} = \bigcap_{n=1}^{\infty} [1 - (1/n), 1].$$

(b) The set $\mathbb{Q} \cap I$ of rational numbers in $[0, 1]$ is countable, i.e., of the form $\{q_1, q_2, \ldots\}$, where each q_i is a different rational number. This set can therefore be written as the union of countably many sets of the form $\{q_i\}$:

$$\mathbb{Q} \cap I = \bigcup_{i=1}^{\infty} \{q_i\}.$$

But a countable union of elements of a σ-algebra (in this instance, Borel sets) belongs to that σ-algebra.

(c) Suppose that $0 \leq a < b \leq 1$. Let k be a positive integer such that $a + (1/k) < b - (1/k)$. Then,

$$(a, b] = \bigcup_{n=k}^{\infty} [a + (1/n), b].$$

Similarly,

$$(a, b) = \bigcup_{n=k}^{\infty} [a + (1/n), b - (1/n)].$$

Note that we can also write (a, b) using complements of Borel sets:

$$(a, b) = \Big([0, a] \cup [b, 1]\Big)^c.$$

From this and part (a), we get

$$(a, b] = (a, b) \cup \{b\}.$$

(d) A subset S of I is said to be **open** if for every $x \in S$, there exists an open interval (a, b) which is contained in S and which contains x. By assumption on S, every $x \in S$ is contained in some interval (a, b) which is contained in S. Using the fact that rational numbers are dense in the reals, we can pick rational numbers q_x and r_x such that $a < q_x < x < r_x < b$. We see that any $x \in S$ is contained in one of the above constructed intervals with rational endpoints. Therefore we can write S as the following union of (possibly uncountably many) open intervals:

$$S = \bigcup_{x \in S} (q_x, r_x).$$

Since there are countably many rationals, the number of such intervals is countable. We conclude that S is a union of countably many intervals (which are Borel sets), and is therefore a Borel set.

$\square$

References

[R] W. Rudin, *Principles of Mathematical Analysis,* McGraw Hill, 1976.

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