MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J | Fall 2008
Problem Set 7 | due 10/29/2008

Readings: Notes for lectures 11-13 (you may skip the proofs in the notes for lecture 11).

Optional additional readings:
Adams & Guillemin, Sections 2.2-2.3, skim Section 2.5.
For a full development of this material, see [W], Sections 5.1-5.9, 6.0-6.3, 6.5, 6.12, 8.0-8.4.

Exercise 1. Show that if $g : \Omega \to [0, \infty]$ satisfies $\int g\, d\mu < \infty$, then $g < \infty$, a.e. (i.e., the set $\{\omega \mid g(\omega) = \infty\}$ has zero measure).

Exercise 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $g : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Let λ be the Lebesgue measure. Let f be a nonnegative measurable function on the real line such that $\int f\, d\lambda = 1$. For any Borel set A, let $\mathbb{P}_1(A) = \int_A f\, d\lambda$. Prove that $\mathbb{P}_1$ is a probability measure.

Exercise 3. (Impulses and Impulse Trains)
Consider the real line, endowed with the Borel σ-field. For any $c \in \mathbb{R}$, we define the Dirac measure ("unit impulse") at c, denoted by δ_c, to be the probability measure that satisfies $\delta_c(c) = 1$. If we "place a Dirac measure" at each integer, we are led to the measure $\mu = \sum_{n=1}^{\infty} \delta_n$, that is, $\mu(A) = \sum_{n=1}^{\infty} \delta_n(A)$, for every Borel set A. (Thus, μ corresponds to an "impulse train" in engineering parlance. It is also a "counting measure", in that it just counts the number of integers in a set A.)

The statements below are all fairly "obvious" properties of impulses. Your task is to provide a formal proof, being careful to use just the definitions above, the general definition of an integral (as a limit using simple functions), and the property that if two functions are equal except on a set of measure zero, then their integrals are equal.

(a) For any nonnegative (not necessarily simple) measurable function $g : \mathbb{R} \to [0, \infty]$, we have $\int g\, d\delta_c = g(c)$.

(b) For any nonnegative (not necessarily simple) measurable function $g : \mathbb{R} \to [0, \infty]$, we have $\int g\, d\mu = \sum_{n=1}^{\infty} g(n)$. (This shows that summation is a special case of integration.)

Exercise 4. (Interchanging summations and limits)
Suppose that the numbers a_{ij}, c_i have the following properties:
(i) For every i, the limit $\lim_{j\to\infty} a_{ij}$ exists;
(ii) For all i, j, we have $|a_{ij}| \leq c_i$;
(iii) $\sum_{i=1}^{\infty} c_i < \infty$.

Use the Dominated Convergence Theorem and a suitable measure to show that

$$\lim_{j\to\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \lim_{j\to\infty} a_{ij}.$$

Exercise 5. (An alternative way of developing integration theory)
We developed in class the standard definition of the integral $\int g\, d\mathbb{P}$ using approximations by simple functions. Let us forget all that and develop a new approach from scratch.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be the real line, endowed with the Borel σ-field, and the Lebesgue measure. We consider the product of these two spaces, and the associated product measure μ on $(\Omega \times \mathbb{R}, \mathcal{F} \times \mathcal{B})$. For any nonnegative random variable X, we define $A_X = \{(\omega, x) \mid 0 \leq x < X(\omega)\}$, and **define** $\mathbb{E}[X] = \mu(A_X)$. (This definition turns out to be equivalent to the standard definition.) The set A is indeed measurable since $A = \bigcup_{q\in\mathbb{Q}} \{(\omega, x) \mid 0 \leq x < q < X(\omega)\}$, and each of the sets in the union are measurable since X is a random variable.

Using the new definition, we would like to verify that various properties of the expectation are easily derived.

Let X, Y be nonnegative random variables. Show the following properties, using just the above definition and basic properties of measures, but no other facts from integration theory.

(a) If we have two nonnegative random variables with $\mathbb{P}(X = Y) = 1$, then $\mathbb{E}[X] = \mathbb{E}[Y]$.

(b) If Y is a nonnegative random variable and $\mathbb{E}[Y] = 0$, then $\mathbb{P}(Y = 0) = 1$.

(c) If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

(d) **(Monotone convergence theorem)** Let X_n be an increasing sequence of nonnegative random variables, whose limit is X. Show that $\lim_{n\to\infty} \mathbb{E}[X_n] \to \mathbb{E}[X]$. *Hint:* This is really easy: use continuity of measures on the sets A_{X_n}.

All this looks pretty simple, so you may wonder why this is not done in most textbooks. The answer is twofold: (i) developing some of the other properties, such as linearity, is not as straightforward; (ii) the construction of the product measure, when carried out rigorously is quite involved.

Exercise 6. Suppose that X is a nonnegative random variable and that $\mathbb{E}[e^{sX}] < \infty$ for all $s \in (-\infty, a]$, where a is a positive number. Let k be a positive integer.

(a) Show that $\mathbb{E}[X^k] < \infty$.

(b) Show that $\mathbb{E}[X^k e^{sX}] < \infty$, for every $s < a$.

(c) Suppose that $h > 0$. Show that $(e^{hX} - 1)/h \leq X e^{hX}$.

(d) Use the DCT to argue that

$$\mathbb{E}[X] = \mathbb{E}\Big[\lim_{h \downarrow 0} \frac{e^{hX} - 1}{h}\Big] = \lim_{h \downarrow 0} \frac{\mathbb{E}[e^{hX}] - 1}{h}.$$

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