MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J — Fall 2008
Problem Set 2 — due 9/15/2008

Readings: Notes from Lectures 2 and 3 (not responsible for the Appendix in Lecture 2). To better understand the material, try the various exercises in the lecture notes.

Optional readings:
(a) Sections 1.4-1.7 of [Grimmett & Stirzaker]
(b) Sections 1.3-1.5 of [Bertsekas & Tsitsiklis] (`http://athenasc.com/Prob-2nd-Ch1.pdf`), including the end of chapter problems.
(c) Chapter 1 of [Williams], for the details of the construction of Lebesgue measure.

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A_1, A_2, \ldots$ be a sequence of $\mathcal{F}$-measurable sets.

(a) Prove that $\mathbb{P}(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mathbb{P}(A_n)$. *Hint:* Recall that $\liminf_n A_n = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} A_n$ is the set of outcomes that belong to all but finitely many A_n, and use various monotonicity and continuity properties of probabilities.

(b) Can you come up with a probability model and a sequence of events for which the inequality in (a) is strict?

(c) Taking for granted the symmetrical inequality

$$\mathbb{P}(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mathbb{P}(A_n),$$

show that if $\lim_n A_n = A$, then $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. *Hint:* Recall that $\lim_n A_n = A$ means $A = \liminf_n A_n = \limsup A_n$. Of course, if the sequence of sets A_n were monotonic, this result would be a special case of the continuity properties of probability measures proved in the Lecture 1 notes.

Exercise 2. We have defined the Borel sets in $I = (0, 1]$ to be the σ-field generated by the intervals of the form $(a, b] = \{x \in I \mid a < x \leq b\}$.

(a) Show that an open interval $(a, b) = \{x \in I \mid a < x < b\}$ is a Borel set.

(b) A subset S of I is said to be **open** if for every $x \in S$, there exists an open interval (a, b) which is contained in S and which contains x. (Intuitively, for every $x \in S$, S contains an "open neighborhood" of x.) Show that if S is open and is contained in I, then S is a Borel set.

(c) Show that the σ-field generated by the open sets is the same as the Borel σ-field.

Hint: Express S as a union of intervals with rational endpoints.

Exercise 3. Suppose that the events A_n satisfy $\mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$. *Note:* A_n i.o., stands for "A_n occurs infinitely often", or "infinitely many of the A_n occur", or just $\limsup_n A_n$. ***Hint:*** Borel-Cantelli.

Exercise 4. Let A_n be a sequence of independent events with $\mathbb{P}(A_n) < 1$ for all n, and $\mathbb{P}(\cup_n A_n) = 1$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 1$. *Note:* A_n i.o., stands for "A_n occurs infinitely often", or "infinitely many of the A_n occur", or just $\limsup_n A_n$. ***Hint:*** Borel-Cantelli.

Exercise 5. Consider an infinite number of independent tosses of a coin. Each toss has probability p, with $0 < p < 1$, of resulting in heads. We say that "a run of length k occurs" if the sequence of heads and tails obtained includes k consecutive heads. Prove that the probability of the following event is 1:

"for every $k > 0$, a run of length k occurs"

(Your proof need not be extremely detailed, but should involve precise mathematical statements.)

Exercise 6. Suppose that A, B, and C are independent events. Use the definition of independence to show that A and $B \cup C$ are independent.

Exercise 7. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let A be an event (element of $\mathcal{F}$). Let $\mathcal{G}$ be the set of all events that are independent from A. Show that $\mathcal{G}$ need not be a σ-field.

Exercise 8. Let A, B, $A_1, A_2, \ldots$ be events. Suppose that for each k, we have $A_k \subseteq A_{k+1}$, and that B is independent of A_k. Let $A = \cup_{k=0}^{\infty} A_k$. Show that B is independent of A.

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