Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 STOCHASTIC PROCESSES, DETECTION AND ESTIMATION

Problem Set 3
Spring 2004

Issued: Thursday, February 19, 2004   **Due:** Thursday, February 26, 2004

Reading: This problem set: Chapter 2, Chapter 3 through Section 3.2.4
Next: Sections 1.7, 3.2.5, 3.3.1, 3.3.2

Problem 3.1
Suppose x and y are the random variables from Problem Set 2, problem 2.4. Their joint density, depicted again below for convenience, is constant in the shaded area and 0 elsewhere.
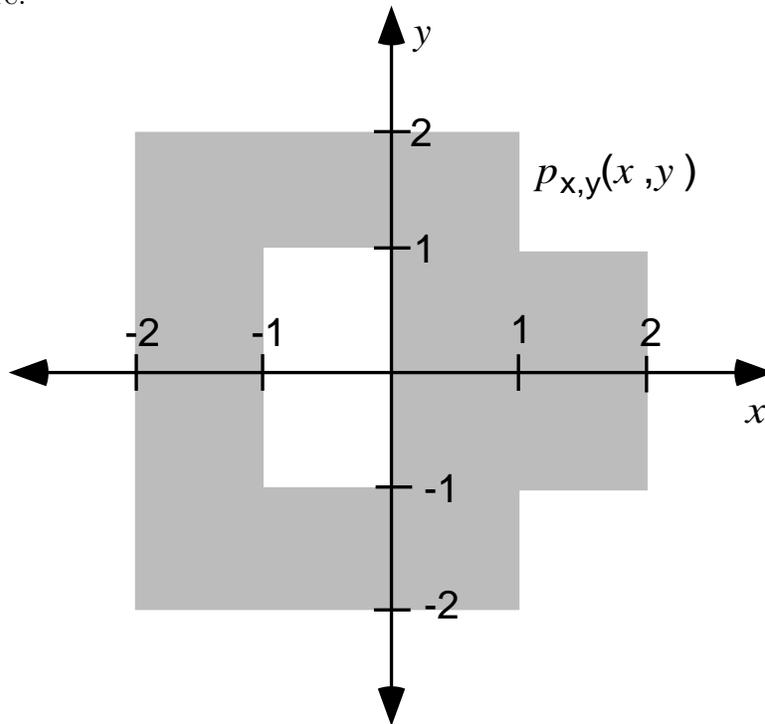
(a) In the (P_D, P_F) plane sketch the operating characeristic of the likelihood ratio test for this problem. Also, indicate on this plot the region consisting of every (P_D, P_F) value that can be achieved using *some* decision rule.

(b) Is the point corresponding to $P_D = \frac{2}{3}$, $P_F = \frac{5}{6}$ in this region? If so, describe a test that achieves this value. If not, explain.

Problem 3.2

We observe a random variable y and have two hypotheses, H_0 and H_1, for its probability density. In particular, the probability densities for y under each of these two hypotheses are depicted below:

$$p_{y|H_0}(y|H_0) = 1, 0 \leq y \leq 1 \qquad\qquad p_{y|H_1}(y|H_1) = 3y^2, 0 \leq y \leq 1$$
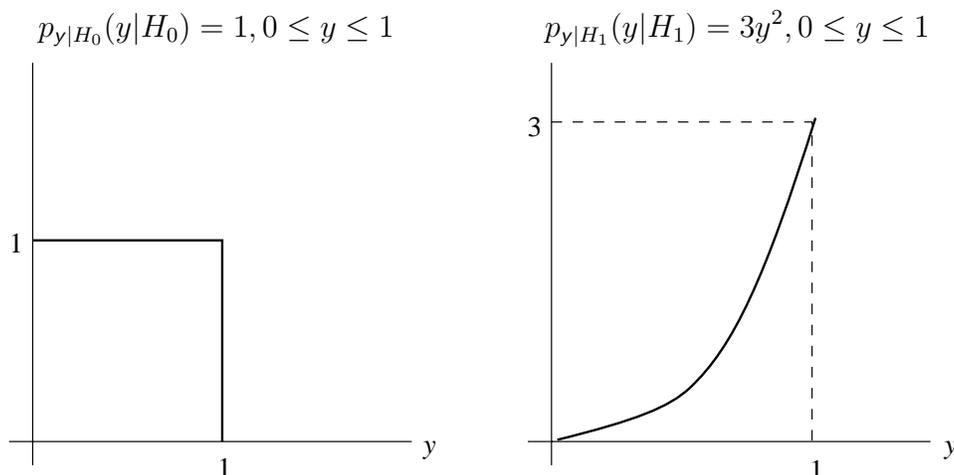
(a) Find the decision rule that maximize P_D subject to the constraint that $P_F \leq \frac{1}{4}$.

(b) Determine the value of P_D for the decision rule specified in part (a).

Problem 3.3

Let k denote the uptime of a communications link in days. Given that the link is functioning at the beginning of a particular day there is probability q that it will go down that day. Thus, the uptime of the link k (in days) obeys a geometric distribution. It is known that q is one of two values: q_0 or q_1. By observing the actual uptime of the link in an experiment, we would like to determine which of the two values of q applies. The hypotheses are then:

$$H_0: \quad p_{k|H}[k|H_0] = \Pr[k = k|H = H_0] = q_0(1 - q_0)^k \quad k = 0, 1, 2, \cdots$$
$$H_1: \quad p_{k|H}[k|H_1] = \Pr[k = k|H = H_1] = q_1(1 - q_1)^k \quad k = 0, 1, 2, \cdots.$$

Suppose that $q_0 = 1/2$, $q_1 = 1/4$, and we observe $k = k$ in our experiment.

(a) Let $\Pr[H = H_0] = \Pr[H = H_1] = \frac{1}{2}$. Find the decision rule $\hat{H}(k)$ corresponding to the minimum probability of error rule.

(b) Make an approximate labeled sketch of the operating characteristics for the likelihood-ratio test (LRT).

(c) Find the decision rule $\hat{H}(k)$ that maximizes $P_D = \Pr\left[\hat{H} = H_1 \mid H = H_1\right]$, subject to the constraint that $P_F \le 0.01$, where $P_F = \Pr\left[\hat{H} = H_1 \mid H = H_0\right]$. What is the corresponding value of P_D?

Problem 3.4 (practice)

A 3-dimensional random vector $\mathbf{y}$ is observed, and we know that one of the three hypotheses is true:

$$
\begin{aligned}
H_1 : & \quad \mathbf{y} = \mathbf{m}_1 + \mathbf{w} \\
H_2 : & \quad \mathbf{y} = \mathbf{m}_2 + \mathbf{w} \\
H_3 : & \quad \mathbf{y} = \mathbf{m}_3 + \mathbf{w},
\end{aligned}
$$

where

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad
\mathbf{m}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad
\mathbf{m}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad
\mathbf{m}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
$$

and $\mathbf{w}$ is a zero-mean Gaussian vector with covariance matrix $\sigma^2 \mathbf{I}$.

(a) Let

$$
\pi(\mathbf{y}) = \begin{bmatrix} \Pr\left[H = H_1 \mid \mathbf{y} = \mathbf{y}\right] \\ \Pr\left[H = H_2 \mid \mathbf{y} = \mathbf{y}\right] \\ \Pr\left[H = H_3 \mid \mathbf{y} = \mathbf{y}\right] \end{bmatrix} = \begin{bmatrix} \pi_1(\mathbf{y}) \\ \pi_2(\mathbf{y}) \\ \pi_3(\mathbf{y}) \end{bmatrix},
$$

and suppose that the Bayes costs are

$$
C_{11} = C_{22} = C_{33} = 0, \qquad C_{12} = C_{21} = 1, \qquad C_{13} = C_{31} = C_{23} = C_{32} = 2.
$$

   (i) Specify the optimum decision rule in terms of $\pi_1(\mathbf{y})$, $\pi_2(\mathbf{y})$, and $\pi_3(\mathbf{y})$.

   (ii) Recalling that $\pi_1 + \pi_2 + \pi_3 = 1$, express this rule completely in terms of π_1 and π_2, and sketch the decision regions in the (π_1, π_2) plane.

(b) Suppose that the three hypotheses are equally likely a priori and that the Bayes costs are

$$
C_{ij} = 1 - \delta_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}.
$$

   Show that the optimum decision rule can be specified in terms of the pair of sufficient statistics

$$
\begin{aligned}
\ell_2(\mathbf{y}) &= y_2 - y_1, \\
\ell_3(\mathbf{y}) &= y_3 - y_1.
\end{aligned}
$$

Hint: To begin, see if you can specify the optimum decision rules in terms of

$$L_i(\mathbf{y}) = \frac{p_{\mathbf{y}|H}(\mathbf{y}|H_i)}{p_{\mathbf{y}|H}(\mathbf{y}|H_1)}, \qquad \text{for } i = 2, 3.$$

Problem 3.5

A former 6.432 student, skilled in the methods of hypothesis testing, came upon a professional gambler who specialized in betting on flips of a coin. Our friend knew that there were three possibilities for the coin the gambler chose:

* The coin is fair. That is, each flip of the coin is independent of all other tosses and has an equal probability of coming up heads or tails.

* The coin is biased towards heads. Specifically, while successive flips of the coin are independent, the probability that any individual flip comes up heads is $3/4$.

* The successive tosses of the coin are *not* independent. Specifically, while the marginal probability of head (or tails) on any given flip is $1/2$, the probability that the next flip yields the same result as the preceding one is only $1/4$. For example, the probability that the next flip comes up heads given that the current flip is a head is $1/4$.

What our friend plans to do is observe two flips of the gambler's coin and then to make a decision among these three possibilities regarding the type of coin the gambler is using.

(a) Assume that our 6.432 expert's prior assessment is that each of these three possibilities is equally likely and that she views as equally bad any error she might make (i.e. she simply wants to minimize the probability of error). Determine the best decision she can make for each of the possible outcomes of the gambler's pair of coin flips.

(b) Determine the probability of error associated with the decision rule determined in part (a).

Problem 3.6

Suppose x and y are random variables. Their joint density, depicted below, is constant in the shaded area and zero elsewhere.
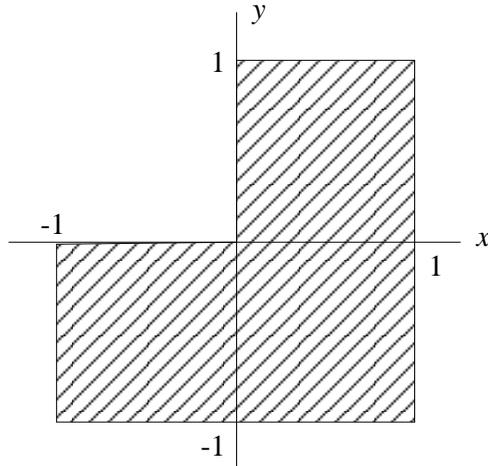
(a) Determine $\hat{x}_{\mathrm{BLS}}(y)$, the Bayes Least Squares estimate of x given y.

(b) Determine λ_{BLS}, the error variance associated with your estimator in (a).

(c) Consider the following "modified least squares" cost function (corresponding to the cost of estimating x as $\hat{x}$):

$$C(x, \hat{x}) = \begin{cases} (x - \hat{x})^2, & x < 0 \\ K(x - \hat{x})^2, & x > 0 \end{cases}$$

where $K > 1$ is a constant.

Determine $\hat{x}_{\mathrm{MLS}}(y)$, the associated Bayes estimate of x for this cost criterion.

(d) Give a brief intuitive explanation for why your answers to (a) and (c) are either the same or different.

Problem 3.7

Let y be a discrete-valued random variable taking on nonnegative integer values, $y = 0, 1, 2, \ldots$, and let x be a continuous-valued random variable. Suppose that the conditional probability mass function for y given x is given by a Poisson density

$$Pr[y|x = x] = \frac{x^y}{y!} e^{-x}, \quad y = 0, 1, 2, \ldots$$

and suppose that x has the probability density

$$p_x(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Compute $\hat{x}_{MAP}(y)$, the MAP estimate of x based on y.

Problem 3.8 (practice)
When a light beam of known intensity v is shined on a photomultiplier the number of photocounts observed in an interval of duration T is a Poisson random variable with mean αv, where $\alpha > 0$ is a known constant. Suppose we shine a light beam of intensity v (v is a random variable) on a photomultiplier and measure

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix}
$$

where y_i is the number of counts in the ith of a set of L non-overlapping intervals each of duration T. Conditioned on v, the numbers of photocounts observed in non-overlapping intervals are statistically independent random variables. For each $1 \le i \le L$,

$$
\Pr\left[y_i = k | v = v \right] = \frac{(\alpha v)^k e^{-\alpha v}}{k!}, \qquad k = 0, 1, 2, \cdots .
$$

(a) Suppose $p_v(v) = \bar{v}^{-1} e^{-v/\bar{v}} u(v)$. Determine $\hat{v}_{\mathrm{BLS}}$, the Bayes least-squares estimate of v, and the resulting mean-square estimation error.

You may find the following identity useful:

$$
\int_0^\infty x^k e^{-ax} \, dx = \frac{k!}{a^{k+1}}
$$

(b) As a check on your answer to part (a), verify that when $\bar{v} \to \infty$, the estimate of part (a) reduces to

$$
\hat{v}_{\mathrm{BLS}} \to \frac{1}{\alpha L} \left(\sum_{i=1}^L y_i + 1 \right).
$$