MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.341 Discrete-Time Signal Processing
Fall 2005

FINAL EXAM

Friday, December 16, 2005 Walker (50-340) 1:30pm–4:30pm

- This is a closed book exam, but **three** $8\frac{1}{2}'' \times 11''$ **handwritten** sheets of notes (both sides) are allowed.
- Calculators are not allowed.
- Make sure you have all **24** numbered pages of this exam.
- There are **9** problems on the exam.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and **show all relevant work**.
- Please be neat—we can not grade what we can not decipher.
- Only this exam booklet is to be handed in. You may want to work things through on scratch paper first, and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.
- We will again be using the EGRMU grading strategy. This strategy focuses on your level of understanding of the material associated with each problem. Specifically, when we grade each part of a problem we will do our best to assess, from your work, your level of understanding.
- **Graded Exams and Final Course Grade:**

  Graded exams, graded Project IIs, and final course grades can be picked up from Eric Strattman (in 36-615 or 36-680, depending on the time of day) on or after WEDNESDAY morning, December 21. If you would like your graded exam and project mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We will use the envelope as is, so please be sure to address it properly and with enough postage. We guarantee that we will put it into the proper mailbox, but we can not guarantee anything beyond that.

OUT OF CONSIDERATION FOR THE 6.341 STAFF, PLEASE DO NOT ASK FOR GRADES BY PHONE OR EMAIL.

THIS PAGE IS INTENTIONALLY LEFT BLANK. YOU CAN USE IT AS SCRATCH PAPER, BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING.

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NAME: ______________________________

Problem	Grade	Points	Grader
1 (a)		/5	
1 (b)		/5	
2 (a)		/6	
2 (b)		/6	
3 (a)		/4	
3 (b)		/4	
3 (c)		/4	
4 (a)		/3	
4 (b)		/3	
4 (c)		/4	
4 (d)		/5	
5 (a)		/6	
5 (b)		/6	
6		/10	
7 (a)		/6	
7 (b)		/6	
8		/8	
9		/9	
Total		/100	

Problem 1 (10%)

[5%] (a) $x[n]$ is a real-valued, causal sequence with discrete-time Fourier transform $X(e^{j\omega})$. Determine a choice for $x[n]$ if the imaginary part of $X(e^{j\omega})$ is given by:

$$\mathrm{Im}\{X(e^{j\omega})\} = 3\sin(2\omega) - 2\sin(3\omega)$$

Work to be looked at and answer:

[5%] (b) $y_r[n]$ is a real-valued sequence with discrete-time Fourier transform $Y_r(e^{j\omega})$. The sequences $y_r[n]$ and $y_i[n]$ in Figure 1-1 are interpreted as the real and imaginary parts of a complex sequence $y[n]$, i.e. $y[n] = y_r[n] + jy_i[n]$.

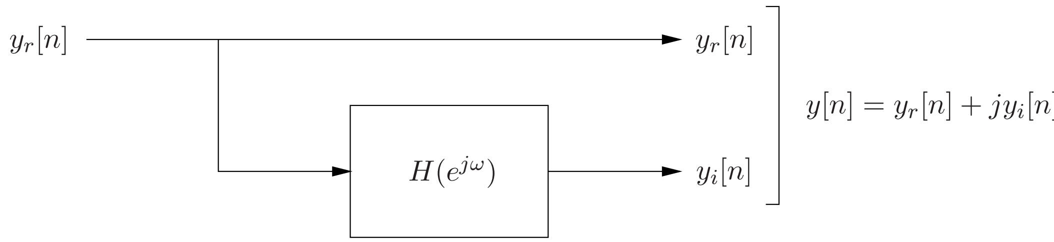


Figure 1-1: System for obtaining $y[n]$ from $y_r[n]$.

Determine a choice for $H(e^{j\omega})$ in Figure 1-1 so that $Y(e^{j\omega})$ is $Y_r(e^{j\omega})$ for *negative* frequencies and zero for *positive* frequencies between $-\pi$ and π, i.e.

$$Y(e^{j\omega}) = \begin{cases} Y_r(e^{j\omega}), & -\pi < \omega < 0 \\ 0, & 0 < \omega < \pi \end{cases}$$

Work to be looked at and answer:

Problem 2 (12%)

Consider the system shown in Figure 2-1, with $H_1(e^{j\omega})$ and $H_2(j\Omega)$ as depicted in Figure 2-2.

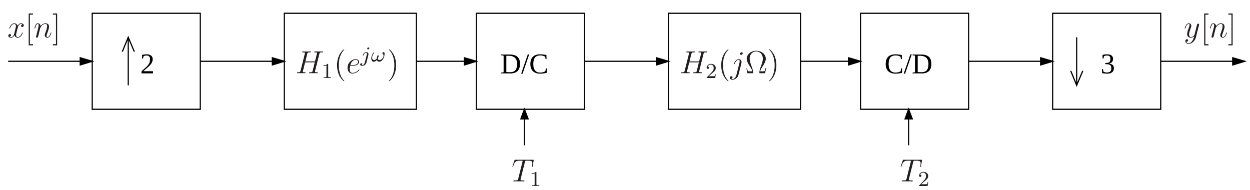


Figure 2-1: System for calculating $y[n]$ from $x[n]$.

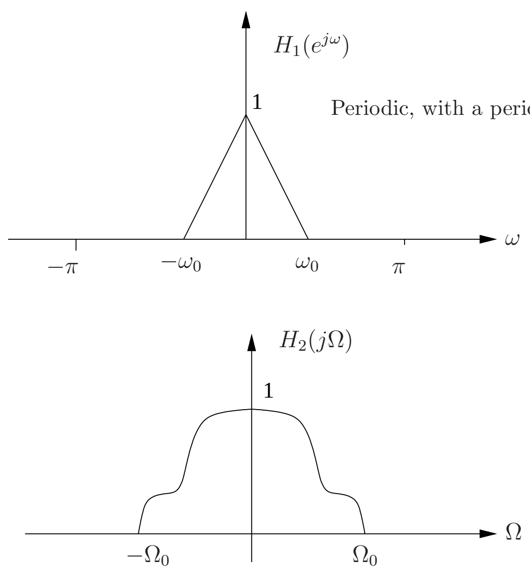


Figure 2-2: Frequency responses of discrete-time LTI filter H_1 and continuous-time LTI filter H_2.

[6%] (a) If $T_1 = T_2 = 10^{-4}$ s and $\Omega_0 = \frac{\pi}{4T_1}$, is there a choice of $\omega_0 > 0$ for which the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is a discrete-time LTI system? If so, specify at least one non-zero value of ω_0 for which the system is LTI. Otherwise explain why the system cannot be LTI.

Work to be looked at and answer:

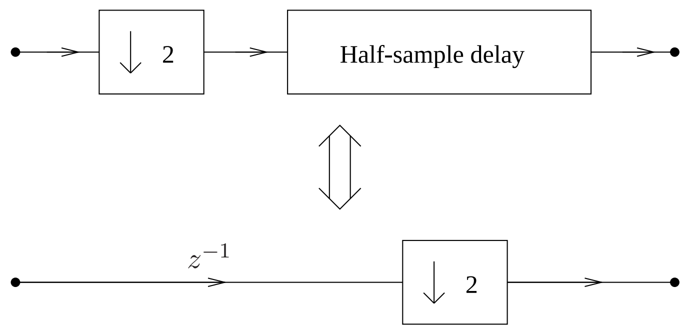
[6%] (b) For this part, assume that $T_1 = 10^{-4}$ s and $\omega_0 = \pi$. Determine the most general conditions on $\Omega_0 > 0$ and T_2, if any, so that the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is an LTI system.

Work to be looked at and answer:

Problem 3 (12%)

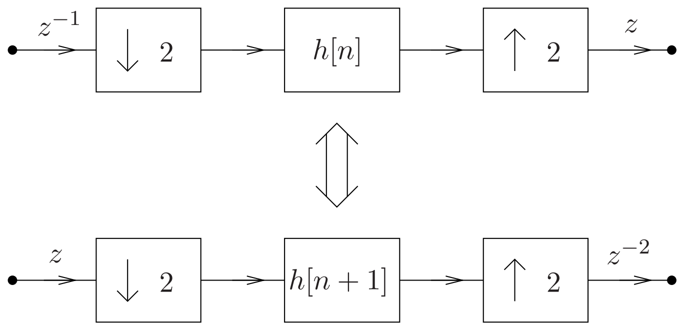
The following are three proposed identites involving compressors and expanders. For each, state whether or not the proposed identity is valid. If your answer is that it is valid, explicitly show why. (In doing this you may make use of the known identities on page 11.) If your answer is no, explicitly give a simple counterexample.

[4%] (a) **Proposed identity (a):**



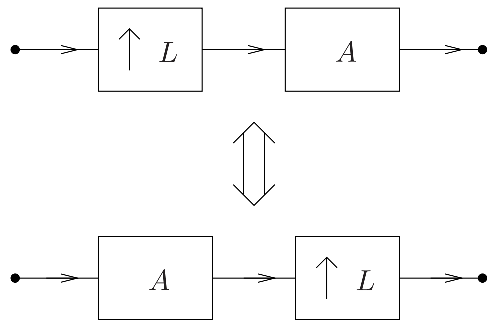
Work to be looked at and answer:

[4%] (b) **Proposed identity (b):**



Work to be looked at and answer:

[4%] (c) **Proposed identity (c):**



where L is a positive integer, and A is defined in terms of $X(e^{j\omega})$ and $Y(e^{j\omega})$ (the respective DTFTs of A's input and output) as:

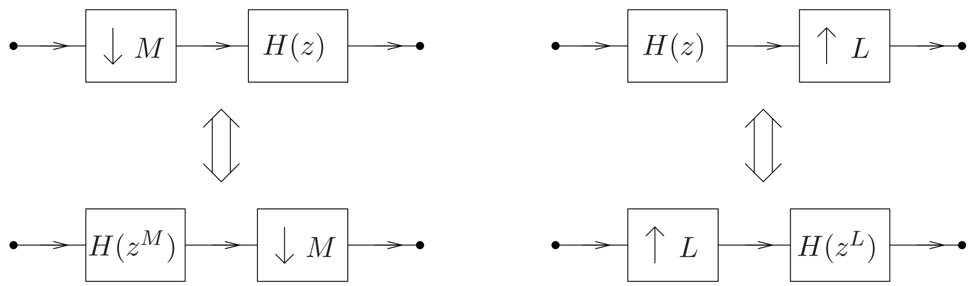
$$x[n] \longrightarrow \boxed{A} \longrightarrow y[n]$$

$$Y(e^{j\omega}) = \left(X(e^{j\omega})\right)^L$$

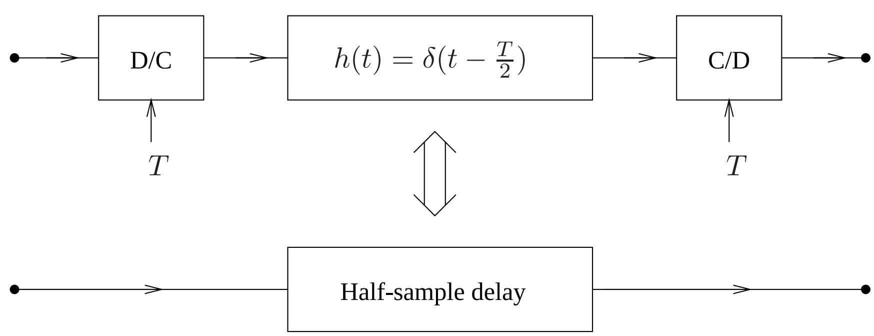
Work to be looked at and answer:

Correct identities you may refer to without proof

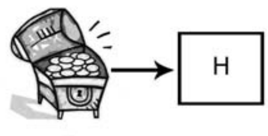
Noble identities:



Half-sample delay:



Problem 4 (15%)



We find in a treasure chest a zero-phase FIR filter $h[n]$ with associated DTFT $H(e^{j\omega})$, shown in Figure 4-1.

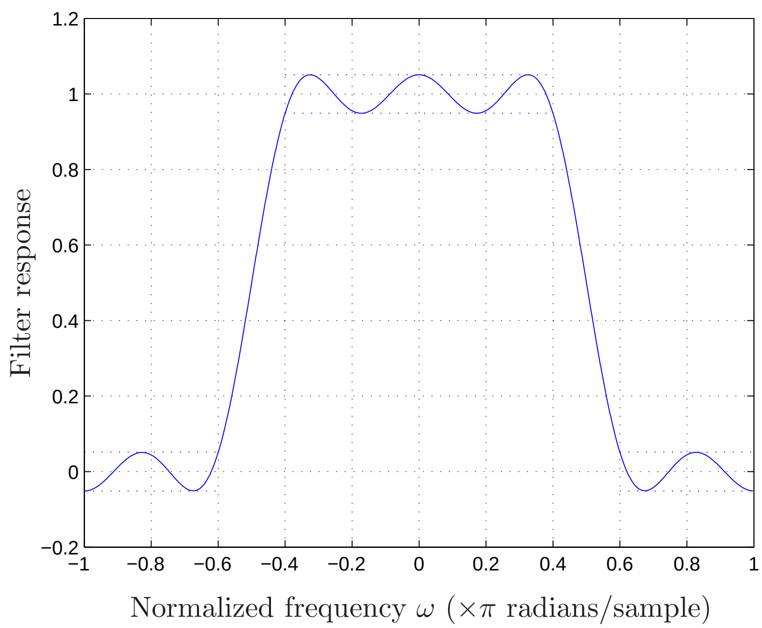


Figure 4-1: Plot of $H(e^{j\omega})$ from $-\pi \leq \omega \leq \pi$.

The filter is known to have been designed using the Parks-McClellan (PM) algorithm, as summarized on page 15 of this exam. The input parameters to the PM algorithm are known to have been:

- Passband edge ω_p: 0.4π
- Stopband edge ω_s: 0.6π
- Ideal passband gain G_p: 1
- Ideal stopband gain G_s: 0
- Error weighting function $W(\omega) = 1$

The value of the input parameter N to the algorithm is not known.

Also along with the plot is a fortune in gold, to be claimed by whoever can reproduce the filter with the Parks-McClellan algorithm for these specifications and with an appropriate value of N. Multiple winners share the gold. You are the sole referee and judge.

Two entries have been submitted, each with a *different* associated value for the input parameter N to the algorithm.

- **Entry 1:** $N = N_1$
- **Entry 2:** $N = N_2 > N_1$

Both entrants claim to have obtained the required filter using exactly the same Parks-McClellan algorithm and input parameters, *except* for the value of N.

After inspecting both entries, you determine that they both have DTFTs identical to Figure 4-1, so you deem both of them winners.

[3%] (a) What are possible values for N_1?

Work to be looked at and answer:

[3%] (b) What are possible values for $N_2 > N_1$?

Work to be looked at and answer:

[4%] (c) Are the impulse responses $h_1[n]$ and $h_2[n]$ of the two filters submitted by entrants 1 and 2 identical?

Work to be looked at and answer:

[5%] (d) Both entrants claim that there can only be *one* winner, since the alternation theorem requires "*uniqueness* of the rth-order polynomial." If your answer to (c) is yes, explain why the alternation theorem is not violated. If your answer is no, show how the two filters, $h_1[n]$ and $h_2[n]$ respectively, relate.

Work to be looked at and answer:

The Parks-McClellan algorithm for zero-phase lowpass filter design:

The algorithm for approximating a lowpass design with a zero-phase PM filter takes as input parameters the following:

- Passband edge frequency ω_p
- Stopband edge frequency ω_s
- Ideal passband gain G_p
- Ideal stopband gain G_s
- Error weighting function $W(\omega)$
- Length N of the filter response $h[n]$, where

$$h[n] = 0 \quad \text{for} \quad |n| > \frac{N-1}{2} \ ,$$

  and N must be odd.

The algorithm returns a filter impulse response which satisfies the alternation theorem, stated below.

Alternation theorem: Let F_P denote the closed subset of the disjoint union of closed subsets of the real axis x. Then

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

is an rth-order polynomial. Also, $D_P(x)$ denotes a given desired function of x that is continuous on F_P; $W_P(x)$ is a positive function, continuous on F_P, and

$$E_P(x) = W_P(x)\,[D_P(x) - P(x)]$$

is the weighted error. The maximum error is defined as

$$\|E\| = \max_{x \in F_P} |E_P(x)| \ .$$

A necessary and sufficient condition that $P(x)$ be the unique rth-order polynomial that minimizes $\|E\|$ is that $E_P(x)$ exhibit *at least* $(r+2)$ alternations; i.e., there must exist at least $(r+2)$ values x_i in F_P such that $x_1 < x_2 < \cdots < x_{r+2}$ and such that $E_P(x_i) = -E_P(x_{i+1}) = \pm\|E\|$ for $i = 1, 2, \ldots, (r+1)$.

Problem 5 (12%)

[6%] (a) $X(e^{j\omega})$ is the DTFT of the discrete-time signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Find a length-5 sequence $g[n]$ whose 5-point DFT $G[k]$ represents samples of the DTFT of $x[n]$, i.e.

$$g[n] = 0 \text{ for } n < 0, \quad n > 4$$

and

$$G[k] = X(e^{j\frac{2\pi k}{5}}) \text{ for } k = 0, 1, \ldots, 4.$$

Work to be looked at and answer:

[6%] (b) Let $w[n]$ be a sequence that is **strictly non-zero** for $0 \le n \le 9$ and zero elsewhere, i.e.

$$\begin{aligned} &w[n] \neq 0, \quad 0 \le n \le 9 \\ &w[n] = 0 \quad \text{otherwise} \end{aligned}$$

Determine a choice for $w[n]$ such that its DTFT $W(e^{j\omega})$ is equal to $X(e^{j\omega})$ at the frequencies $\omega = \frac{2\pi k}{5},\ k = 0, 1, \ldots, 4$, i.e.

$$W(e^{j\frac{2\pi k}{5}}) = X(e^{j\frac{2\pi k}{5}}) \text{ for } k = 0, 1, \ldots, 4.$$

Work to be looked at and answer:

Problem 6 (10%)

A system for the discrete-time spectral analysis of continuous-time signals is shown in Figure 6-1.

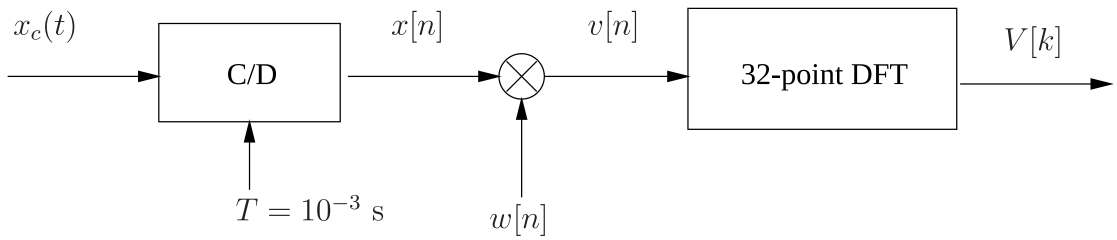


Figure 6-1: Spectral analysis system.

$w[n]$ is a rectangular window of length 32:

$$w[n] = \begin{cases} (1/32), & 0 \leq n \leq 31 \\ 0, & \text{otherwise} \end{cases}$$

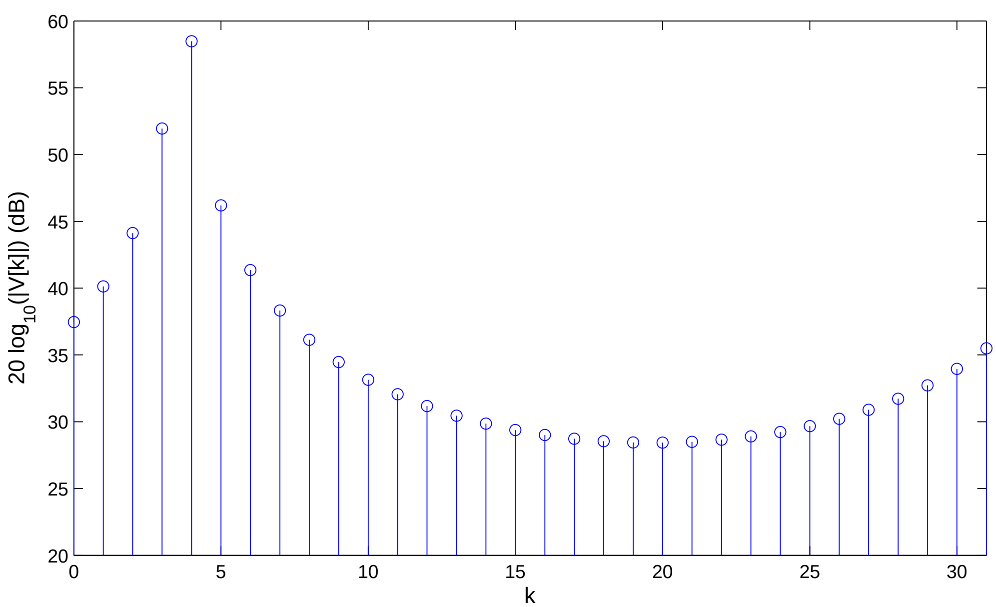


Figure 6-2: Output $|V[k]|$ in dB

Listed below are ten signals, at least one of which was the input $x_c(t)$. Indicate which signal(s) could have been the input $x_c(t)$ which produced the plot of $|V[k]|$ shown in dB units in Figure 6-2. As always, provide reasoning for your choice(s).

$$x_1(t) = 1000\cos(230\pi t) \qquad x_6(t) = 1000e^{j(250)\pi t}$$
$$x_2(t) = 1000\cos(115\pi t) \qquad x_7(t) = 10\cos(250\pi t)$$
$$x_3(t) = 10e^{j(460)\pi t} \qquad x_8(t) = 1000\cos(218.75\pi t)$$
$$x_4(t) = 1000e^{j(230)\pi t} \qquad x_9(t) = 10e^{j(200)\pi t}$$
$$x_5(t) = 10e^{j(230)\pi t} \qquad x_{10}(t) = 1000e^{j(187.5)\pi t}$$

Work to be looked at and answer:

Problem 7 (12%)

$x[n]$ is a finite-length sequence of length 1024, i.e.

$$x[n] = 0 \quad \text{for } n < 0,\ n > 1023.$$

The autocorrelation of $x[n]$ is defined as

$$R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m],$$

and $X_N[k]$ is defined as the N-point DFT of $x[n]$, with $N \geq 1024$.

We are interested in computing $R_{xx}[m]$. A proposed procedure begins by first generating the N-point inverse DFT of $|X_N[k]|^2$ to obtain an N-point sequence $g_N[n]$, i.e.

$$g_N[n] = \text{N-point IDFT} \left\{ |X_N[k]|^2 \right\} \ .$$

[6%] (a) Determine the minimum value of N so that $R_{xx}[m]$ can be obtained from $g_N[n]$. Also specify how you would obtain $R_{xx}[m]$ from $g_N[n]$.

Work to be looked at and answer:

[6%] (b) Determine the minimum value of N so that $R_{xx}[m]$ for $|m| \leq 10$ can be obtained from $g_N[n]$. Also specify how you would obtain these values from $g_N[n]$.

Work to be looked at and answer:

Problem 8 (8%)

A system for examining the spectral content of a signal $x[n]$ is shown in Figure 8-1. The filters $h[n]$ in each channel are identical three-point non-causal FIR filters and have impulse response

$$h[n] = h_0\delta[n] + h_1\delta[n+1] + h_2\delta[n+2].$$

The filter outputs are sampled at $n = 0$ to obtain the sequence $y_k[0]$, $k = 0, 1, 2, 3$.

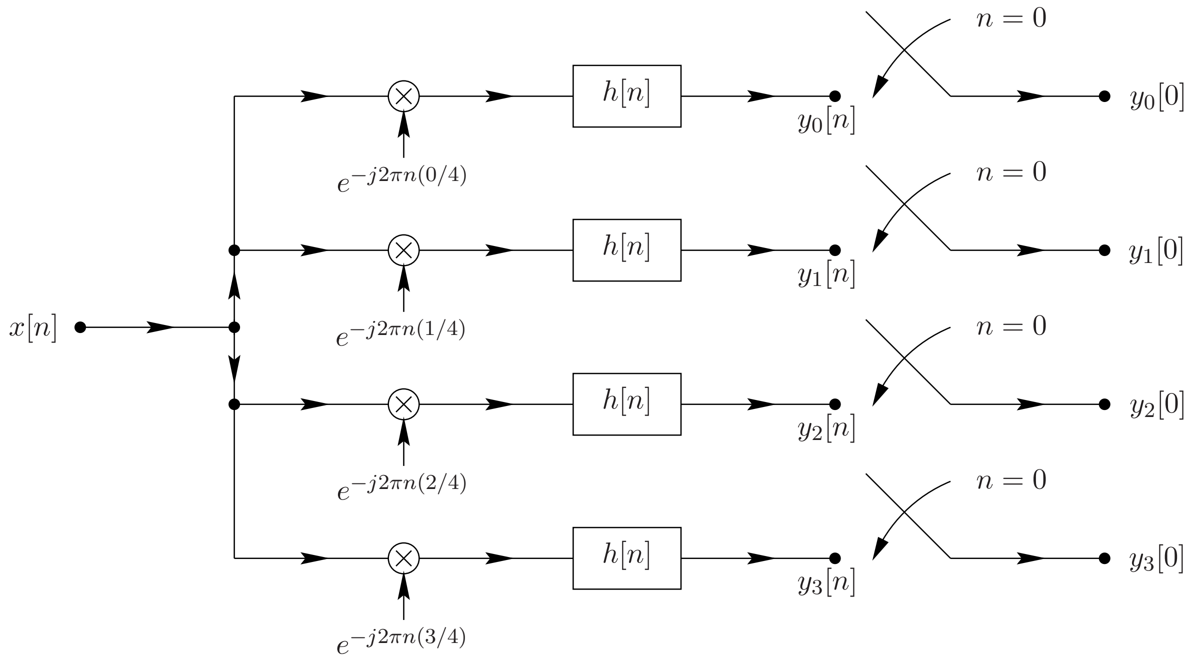


Figure 8-1: Filter bank network.

An alternative to the system in Figure 8-1 has been proposed using a 4-point DFT as shown in Figure 8-2.

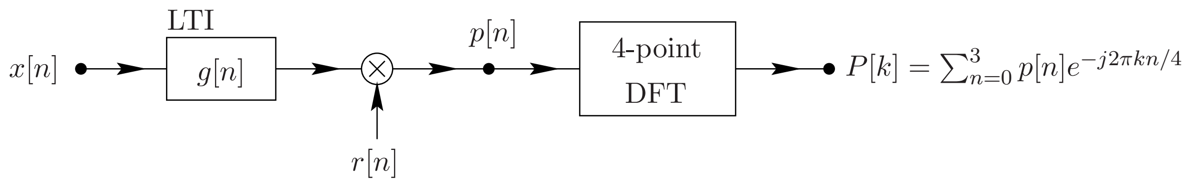


Figure 8-2: Alternative system.

Determine $g[n]$ and $r[n]$ so that $P[k] = y_k[0]$.

Work to be looked at and answer:

Problem 9 (9%)

Consider the system shown in Figure 9-1. The subsystem from $x[n]$ to $y[n]$ is a causal, LTI system implementing the difference equation

$$y[n] = x[n] + ay[n-1].$$

$x[n]$ is a finite length sequence of length 90, i.e.

$$x[n] = 0 \quad \text{for } n < 0 \text{ and } n > 89.$$

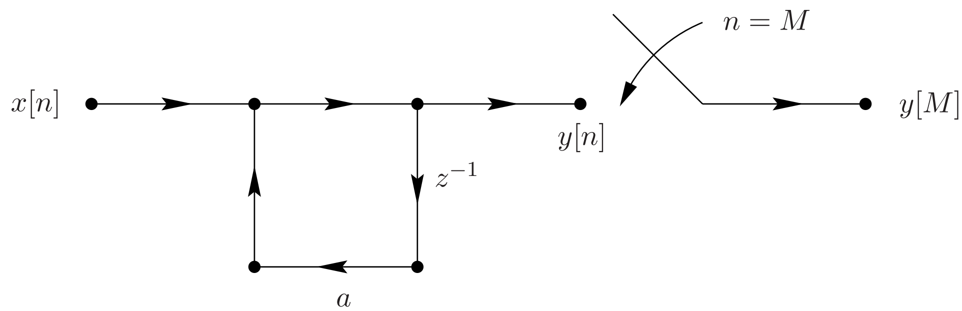


Figure 9-1: System for calculating $y[M]$ from $x[n]$.

Determine a choice for the complex constant a and a choice for the sampling instant M so that

$$y[M] = X(e^{j\omega})\big|_{\omega=2\pi/60}$$

Work to be looked at and answer:

END OF EXAM