MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.341 Discrete-Time Signal Processing
Fall 2005

Solution

FINAL EXAM

Friday, December 16, 2005 Walker (50-340) 1:30pm 4:30pm

- This is a closed book exam, but **three** $8\frac{1}{2}'' \times 11''$ **handwritten** sheets of notes (both sides) are allowed.
- Calculators are not allowed.
- Make sure you have all **24** numbered pages of this exam.
- There are **9** problems on the exam.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and **show all relevant work**.
- Please be neat we can not grade what we can not decipher.
- Only this exam booklet is to be handed in. You may want to work things through on scratch paper first, and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.
- We will again be using the EGRMU grading strategy. This strategy focuses on your level of understanding of the material associated with each problem. Specifically, when we grade each part of a problem we will do our best to assess, from your work, your level of understanding.
- **Graded Exams and Final Course Grade:**

  Graded exams, graded Project IIs, and final course grades can be picked up from Eric Strattman (in 36-615 or 36-680, depending on the time of day) on or after WEDNESDAY morning, December 21. If you would like your graded exam and project mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We will use the envelope as is, so please be sure to address it properly and with enough postage. We guarantee that we will put it into the proper mailbox, but we can not guarantee anything beyond that.

OUT OF CONSIDERATION FOR THE 6.341 STAFF, PLEASE DO NOT ASK FOR GRADES BY PHONE OR EMAIL.

THIS PAGE IS INTENTIONALLY LEFT BLANK. YOU CAN USE IT AS SCRATCH PAPER, BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.341 DISCRETE-TIME SIGNAL PROCESSING
Fall 2005

FINAL EXAM
Friday, December 16, 2005

NAME: 6.341 Staff

Problem	Grade	Points	Grader
1 (a)	E	5 /5	
1 (b)	E	5 /5	
2 (a)	E	6 /6	
2 (b)	E	6 /6	
3 (a)	E	4 /4	
3 (b)	E	4 /4	
3 (c)	E	4 /4	
4 (a)	E	3 /3	
4 (b)	E	3 /3	
4 (c)	E	4 /4	
4 (d)	E	5 /5	
5 (a)	E	6 /6	
5 (b)	E	6 /6	
6	E	10 /10	
7 (a)	E	6 /6	
7 (b)	E	6 /6	
8	E	8 /8	
9	E	9 /9	
Total		100 /100	

Problem 1 (10%)

[5%] (a) $x[n]$ is a real-valued, causal sequence with discrete-time Fourier transform $X(e^{j\omega})$. Determine a choice for $x[n]$ if the imaginary part of $X(e^{j\omega})$ is given by:

$$\mathrm{Im}\{X(e^{j\omega})\} = 3\sin(2\omega) - 2\sin(3\omega)$$

Work to be looked at and answer:

Since $x[n]$ is real, $j\mathrm{Im}\{X(e^{j\omega})\}$ is the Fourier transform of $x_o[n]$, the odd part of $x[n]$.

$$j\mathrm{Im}\{X(e^{j\omega})\} = \cancel{j}\left[\frac{3}{2\cancel{j}}(e^{j2\omega} - e^{-j2\omega}) - \frac{\cancel{2}}{\cancel{2}\cancel{j}}(e^{j3\omega} - e^{-j3\omega})\right]$$

$$= \frac{3}{2}e^{j2\omega} - \frac{3}{2}e^{-j2\omega} - e^{j3\omega} + e^{-j3\omega}$$

$$\longleftrightarrow x_o[n] = \frac{3}{2}\delta[n+2] - \frac{3}{2}\delta[n-2] - \delta[n+3] + \delta[n-3]$$

$x[n]$ is also causal, so it can be recovered by doubling $x_o[n]$ for $n > 0$ and setting it to zero for $n < 0$. The only value that cannot be determined is $x[0]$ so we leave it as arbitrary.

$$\boxed{x[n] = 2\delta[n-3] - 3\delta[n-2] + x[0]\delta[n]}$$

[5%] (b) $y_r[n]$ is a real-valued sequence with discrete-time Fourier transform $Y_r(e^{j\omega})$. The sequences $y_r[n]$ and $y_i[n]$ in Figure 1-1 are interpreted as the real and imaginary parts of a complex sequence $y[n]$, i.e. $y[n] = y_r[n] + jy_i[n]$.

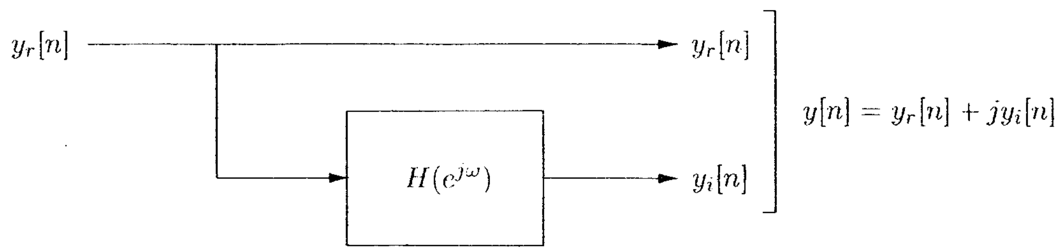


Figure 1-1: System for obtaining $y[n]$ from $y_r[n]$.

Determine a choice for $H(e^{j\omega})$ in Figure 1-1 so that $Y(e^{j\omega})$ is $Y_r(e^{j\omega})$ for *negative* frequencies and zero for *positive* frequencies between $-\pi$ and π, i.e.

$$Y(e^{j\omega}) = \begin{cases} Y_r(e^{j\omega}), & -\pi < \omega < 0 \\ 0, & 0 < \omega < \pi \end{cases}$$

Work to be looked at and answer:

$$Y(e^{j\omega}) = Y_r(e^{j\omega}) + jY_i(e^{j\omega})$$
$$= Y_r(e^{j\omega})\left(1 + jH(e^{j\omega})\right)$$

To satisfy the constraint,

$$1 + jH(e^{j\omega}) = \begin{cases} 1, & -\pi < \omega < 0 \\ 0, & 0 < \omega < \pi \end{cases}$$

$$\boxed{H(e^{j\omega}) = \begin{cases} 0, & -\pi < \omega < 0 \\ j, & 0 < \omega < \pi \end{cases}}$$

Problem 2 (12%)

Consider the system shown in Figure 2-1, with $H_1(e^{j\omega})$ and $H_2(j\Omega)$ as depicted in Figure 2-2.

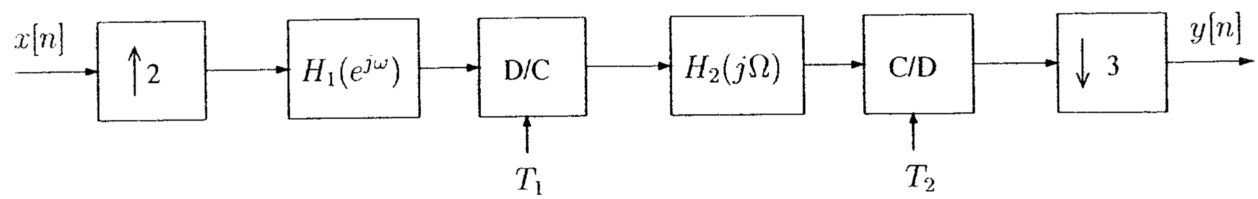


Figure 2-1: System for calculating $y[n]$ from $x[n]$.

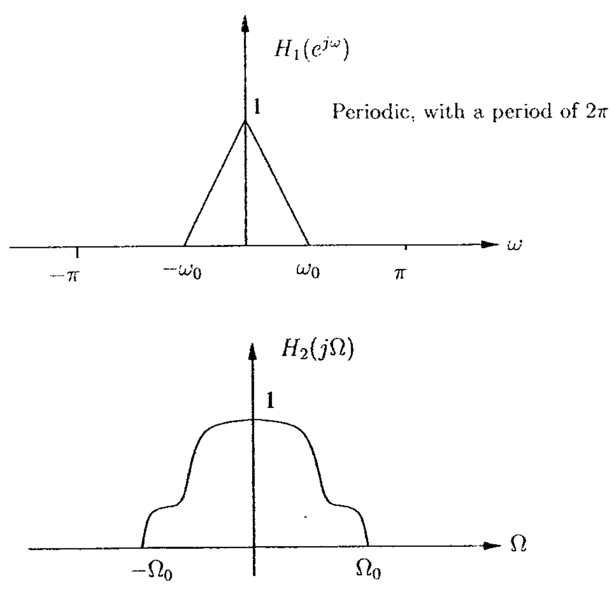


Figure 2-2: Frequency responses of discrete-time LTI filter H_1 and continuous-time LTI filter H_2.

[6%] (a) If $T_1 = T_2 = 10^{-4}$ s and $\Omega_0 = \frac{\pi}{4T_1}$, is there a choice of $\omega_0 > 0$ for which the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is a discrete-time LTI system? If so, specify at least one non-zero value of ω_0 for which the system is LTI. Otherwise explain why the system cannot be LTI.

Work to be looked at and answer:

When $T_1 = T_2$ the D/C converter, CT LTI filter, and C/D can be replaced with a DT LTI filter with frequency response

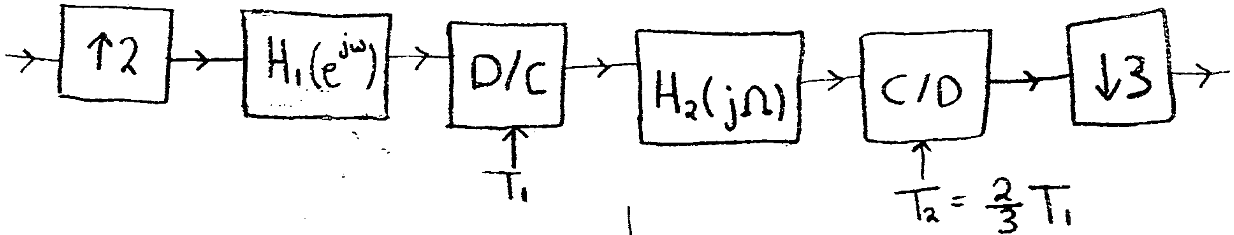
$$H_2'(e^{j\omega}) = H_2\left(j\frac{\omega}{T_1}\right), \quad \text{for } |\omega| < \pi$$

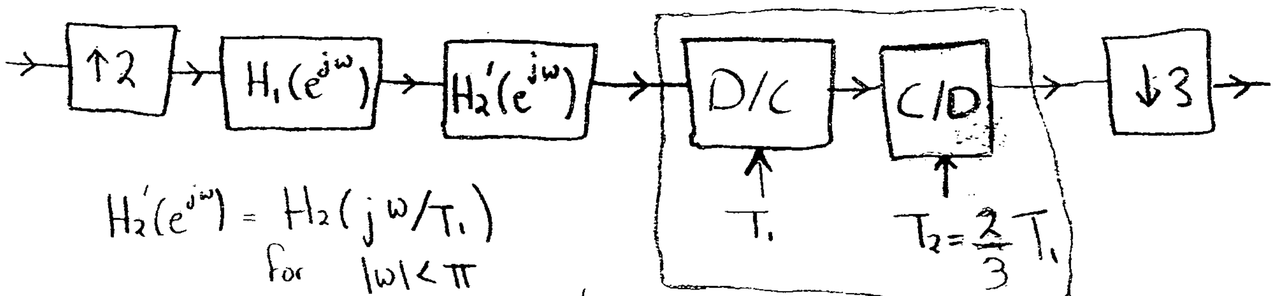
An overall effect of the expander and compressor is to stretch the frequency axis of the input DTFT by a factor of $\frac{3}{2}$, so the system cannot be LTI in this case because the stretching is not undone. It can be verified that the output when $x[n] = g[n-1]$ is not a delay of the output when $x[n] = g[n]$ for all $g[n]$.

[6%] (b) For this part, assume that $T_1 = 10^{-4}$ s and $\omega_0 = \pi$. Determine the most general conditions on $\Omega_0 > 0$ and T_2, if any, so that the overall system from $x[n]$ to $y[n]$ in Figure 2-1 is an LTI system.

Work to be looked at and answer:

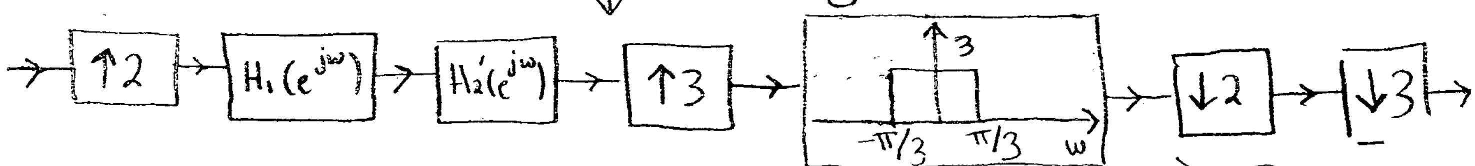
To undo the scaling effect on the frequency axis, we need to set $T_2 = \frac{2}{3}T_1$. With this choice of T_2 the C/D and D/C will implement a sample rate increase by a noninteger factor of $\frac{3}{2}$. The overall system then simplifies to a DT LTI system as shown on the next page, without any additional conditions on Ω_0.



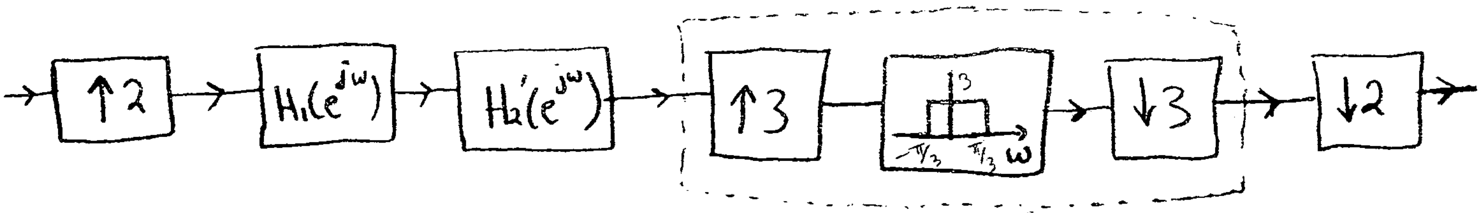


$$H_2'(e^{j\omega}) = H_2(j\omega/T_1)$$
for $|\omega| < \pi$

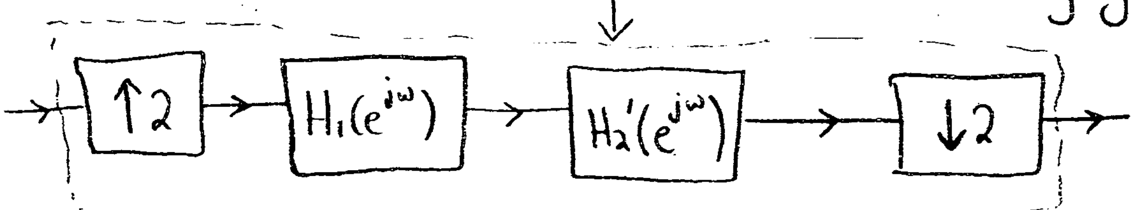
sampling rate converter



switch order



identity system



DT LTI system

frequency response : $\frac{1}{2} H_1(e^{j\omega/2}) H_2'(e^{j\omega/2})$

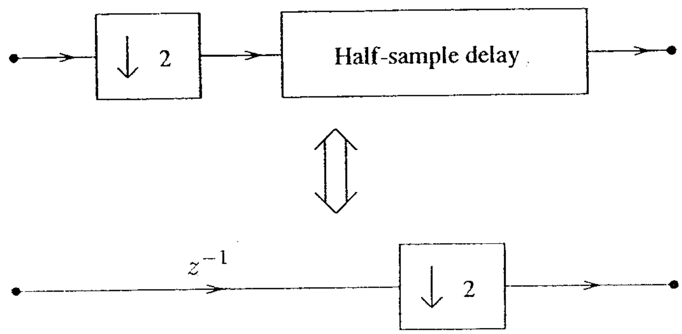
$+ \frac{1}{2} H_1(e^{j(\omega/2 - \pi)}) H_2'(e^{j(\omega/2 - \pi}$)

This page has intentionally been left blank

Problem 3 (12%)

The following are three proposed identites involving compressors and expanders. For each, state whether or not the proposed identity is valid. If your answer is that it is valid, explicitly show why. (In doing this you may make use of the known identities on page 11.) If your answer is no, explicitly give a simple counterexample.

[4%] (a) **Proposed identity (a):**



Work to be looked at and answer:

→ This proposed identity is not valid.
Consider as an input $\delta[n]$.

$\delta[n] \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\text{Half-sample delay}} \rightarrow$

$\Downarrow$

$\delta[n] \rightarrow \boxed{\text{Half-sample delay}} \rightarrow \neq 0$ for some n a_1

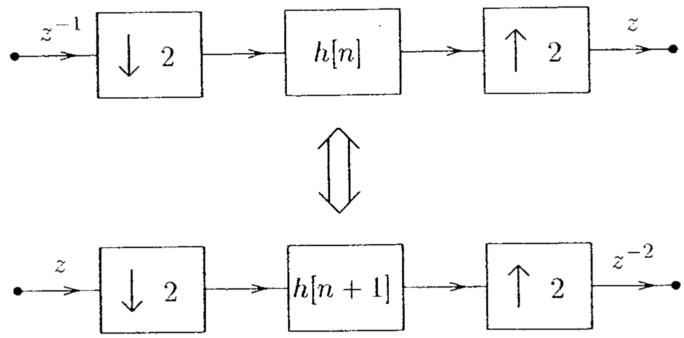
But

$\delta[n] \xrightarrow{z^{-1}} \boxed{\downarrow 2} \rightarrow = 0$ for all n a_2

$a_1 \neq a_2$

Name:____________________________________

[4%] (b) **Proposed identity (b):**



Work to be looked at and answer:

→ This proposed identity is not valid.
Consider as input $\delta[n-1]$, and consider $h[n] = \delta[n-1]$.

$$\delta[n-1] \xrightarrow{z^{-1}} \boxed{\downarrow 2} \to \boxed{\delta[n-1]} \to \boxed{\uparrow 2} \xrightarrow{z}$$

$$\downarrow$$

$$\delta[n-1] \to \boxed{\delta[n-1]} \to \boxed{\uparrow 2} \xrightarrow{z}$$

$$\downarrow$$

$$\delta[n-4] \xrightarrow{z} \; = \; \delta[n-3] \qquad b_1$$

But

$$\delta[n-1] \xrightarrow{z} \boxed{\downarrow 2} \to \boxed{\delta[n]} \to \boxed{\uparrow 2} \xrightarrow{z^{-2}}$$

$$\downarrow$$

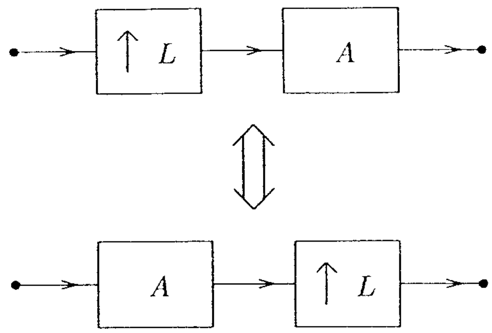
$$\delta[n] \to \boxed{\delta[n]} \to \boxed{\uparrow 2} \xrightarrow{z^{-2}}$$

$$\downarrow$$

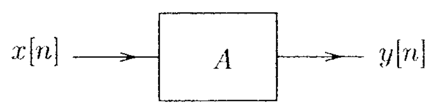
$$\delta[n] \xrightarrow{z^{-2}} \; = \; \delta[n-2] \qquad b_2$$

$$b_1 \neq b_2$$

[4%] (c) **Proposed identity (c):**



where L is a positive integer, and A is defined in terms of $X(e^{j\omega})$ and $Y(e^{j\omega})$ (the respective DTFTs of A's input and output) as:



$$Y(e^{j\omega}) = \left(X(e^{j\omega})\right)^L$$

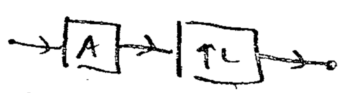
Work to be looked at and answer:

→ This proposed identity is valid. This is demonstrated by looking in the frequency domain.

- Considering first system, and input $v[n]$ (DTFT $V(e^{j\omega})$):

$$V(e^{j\omega}) \rightarrow V(e^{j\omega L}) \rightarrow \left(V(e^{j\omega L})\right)^L \qquad C_1$$

- Now considering second system, and same input $v[n]$.

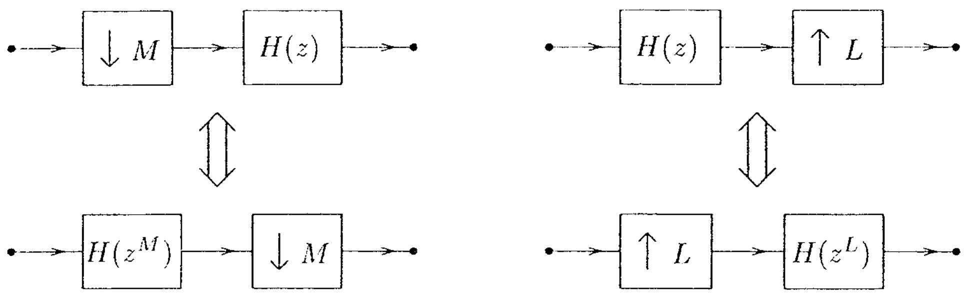


$$V(e^{j\omega}) \rightarrow \left(V(e^{j\omega})\right)^L \rightarrow \left(V(e^{j\omega L})\right)^L \qquad C_2$$

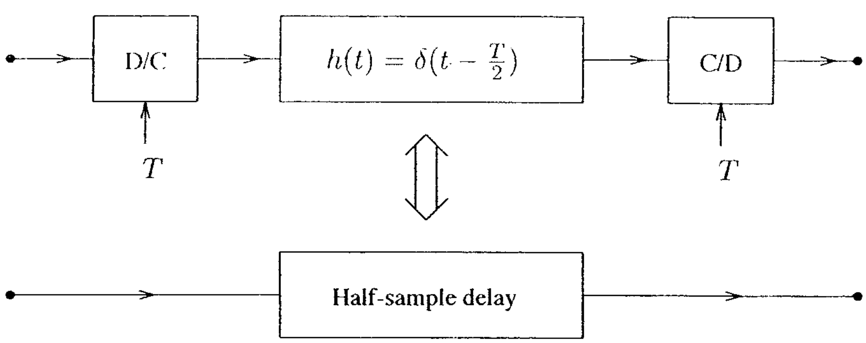
$C_1 = C_2$

Correct identities you may refer to without proof

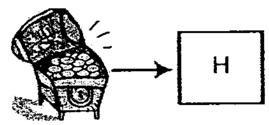
Noble identities:



Half-sample delay:



Problem 4 (15%)



We find in a treasure chest a zero-phase FIR filter $h[n]$ with associated DTFT $H(e^{j\omega})$, shown in Figure 4-1.

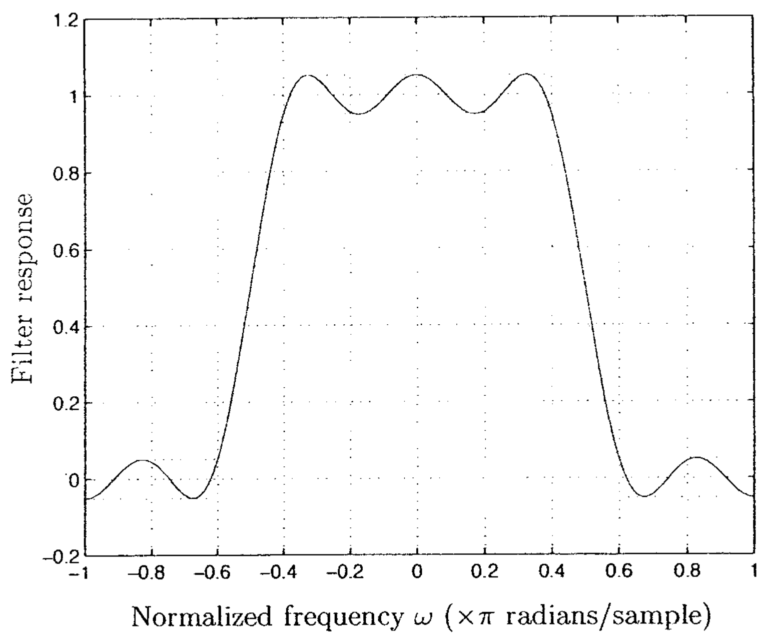


Figure 4-1: Plot of $H(e^{j\omega})$ from $-\pi \leq \omega \leq \pi$.

The filter is known to have been designed using the Parks-McClellan (PM) algorithm, as summarized on page 15 of this exam. The input parameters to the PM algorithm are known to have been:

- Passband edge ω_p: 0.4π
- Stopband edge ω_s: 0.6π
- Ideal passband gain G_p: 1
- Ideal stopband gain G_s: 0
- Error weighting function $W(\omega) = 1$

The value of the input parameter N to the algorithm is not known.

Also along with the plot is a fortune in gold, to be claimed by whoever can reproduce the filter with the Parks-McClellan algorithm for these specifications and with an appropriate value of N. Multiple winners share the gold. You are the sole referee and judge.

Two entries have been submitted, each with a *different* associated value for the input parameter N to the algorithm.

- **Entry 1:** $N = N_1$
- **Entry 2:** $N = N_2 > N_1$

Both entrants claim to have obtained the required filter using exactly the same Parks-McClellan algorithm and input parameters, *except* for the value of N.

After inspecting both entries, you determine that they both have DTFTs identical to Figure 4-1, so you deem both of them winners.

[3%] (a) What are possible values for N_1?

Work to be looked at and answer:

From Fig 4-1, $H(e^{j\omega})$ exhibits 8 alternations of the error ($0 \le \omega \le \pi$) since it is an approximation to an ideal LPF with parameters given on p. 12. Because a lowpass filter designed with P-M has either L+2 or L+3 alternations and because we're told that there is another filter out there which meets the specs for $N_2 > N_1$, we should consider the L+3 case to find the smaller N.

$8 = L+3$ alternations $\Rightarrow L = 5$; $\frac{N-1}{2} = 5 \Rightarrow \boxed{N_1 = 11 \text{ only}}$

[3%] (b) What are possible values for $N_2 > N_1$?

Work to be looked at and answer:

Since there are 8 alternations, L_2 can be no greater than 6. Therefore $\frac{N_2 - 1}{2} \le 6 \Rightarrow N_2 \le 13$. The only other possible value of N for a LPF was found in (a), so $\boxed{N_2 = 13 \text{ only}}$

[4%] (c) Are the impulse responses $h_1[n]$ and $h_2[n]$ of the two filters submitted by entrants 1 and 2 identical?

Work to be looked at and answer:

Yes. Since both have DTFTs identical to Fig. 4-1,

$h_1[n] = h_2[n]$.

[5%] (d) Both entrants claim that there can only be *one* winner, since the alternation theorem requires "*uniqueness* of the rth-order polynomial." If your answer to (c) is yes, explain why the alternation theorem is not violated. If your answer is no, show how the two filters, $h_1[n]$ and $h_2[n]$ respectively, relate.

Work to be looked at and answer:

While the alternation theorem states that for a given r, there is a unique rth-order polynomial which satisfies it, the theorem makes no claim about how this polynomial may or may not relate to a poly. satisfying the alt. thm. for a different r.

It turns out that in this case, the single 5th-order polynomial satisfying the alt. thm. for $r_1 = L_1 = 5$ is identical to the single 6th-order polynomial satisfying the alt. thm. for $r_2 = L_2 = 6$.

The Parks-McClellan algorithm for zero-phase lowpass filter design:

The algorithm for approximating a lowpass design with a zero-phase PM filter takes as input parameters the following:

- Passband edge frequency ω_p
- Stopband edge frequency ω_s
- Ideal passband gain G_p
- Ideal stopband gain G_s
- Error weighting function $W(\omega)$
- Length N of the filter response $h[n]$, where

  $$h[n] = 0 \quad \text{for} \quad |n| > \frac{N-1}{2} \; ,$$

  and N must be odd.

The algorithm returns a filter impulse response which satisfies the alternation theorem, stated below.

Alternation theorem: Let F_P denote the closed subset of the disjoint union of closed subsets of the real axis x. Then

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

is an rth-order polynomial. Also, $D_P(x)$ denotes a given desired function of x that is continuous on F_P; $W_P(x)$ is a positive function, continuous on F_P, and

$$E_P(x) = W_P(x)\,[D_P(x) - P(x)]$$

is the weighted error. The maximum error is defined as

$$\|E\| = \max_{x \in F_P} |E_P(x)| \; .$$

A necessary and sufficient condition that $P(x)$ be the unique rth-order polynomial that minimizes $\|E\|$ is that $E_P(x)$ exhibit *at least* $(r+2)$ alternations; i.e., there must exist at least $(r+2)$ values x_i in F_P such that $x_1 < x_2 < \cdots < x_{r+2}$ and such that $E_P(x_i) = -E_P(x_{i+1}) = \pm\|E\|$ for $i = 1, 2, \ldots, (r+1)$.

Problem 5 (12%)

[6%] (a) $X(e^{j\omega})$ is the DTFT of the discrete-time signal

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Find a length-5 sequence $g[n]$ whose 5-point DFT $G[k]$ represents samples of the DTFT of $x[n]$, i.e.

$$g[n] = 0 \quad \text{for} \quad n < 0, \quad n > 4$$

and

$$G[k] = X(e^{j\frac{2\pi k}{5}}) \text{ for } k = 0, 1, \ldots, 4.$$

Work to be looked at and answer:

To obtain 5 samples of $X(e^{j\omega})$, we need to time-alias $x[n]$ to $0 \leq n < 5$ and take a DFT. Sampling at 5 points in frequency corresponds to periodically replicating $x[n]$ with a period of 5, summing the replicas, and extracting the first 5 points by multiplying with a window.

$$\begin{aligned}
g[n] &= \sum_{m=-\infty}^{\infty} x[n+5m] && \text{for } 0 \leq n < 5 \\
&= \sum_{m=0}^{\infty} x[n+5m] && \text{for } 0 \leq n < 5 \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{n+5m} && \text{for } 0 \leq n < 5 \\
&= \left(\frac{1}{2}\right)^n \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{5m} && \text{for } 0 \leq n < 5 \\
&= \left(\frac{1}{2}\right)^n \left(\frac{1}{1-(1/2)^5}\right) && \text{for } 0 \leq n < 5
\end{aligned}$$

[6%] (b) Let $w[n]$ be a sequence that is **strictly non-zero** for $0 \leq n \leq 9$ and zero elsewhere, i.e.

$$w[n] \neq 0, \quad 0 \leq n \leq 9$$
$$w[n] = 0 \quad \text{otherwise}$$

Determine a choice for $w[n]$ such that its DTFT $W(e^{j\omega})$ is equal to $X(e^{j\omega})$ at the frequencies $\omega = \frac{2\pi k}{5}$, $k = 0, 1, \ldots, 4$, i.e.

$$W(e^{j\frac{2\pi k}{5}}) = X(e^{j\frac{2\pi k}{5}}) \text{ for } k = 0, 1, \ldots, 4.$$

Work to be looked at and answer:

To get the samples of $W(e^{j\omega})$ at 5 frequency points, we need to alias $w[n]$ to 5 points and take its DFT. If

$$w_a[n] = \begin{cases} w[n] + w[n+5] & 0 \leq n < 5 \\ 0 & \text{otherwise} \end{cases}$$

then $w_a[n]$ must be equal to g[n] to have the same DFT. So $w[n]$ must satisfy $w[n] + w[n+5] = g[n]$ for $0 \leq n < 5$.

We can find one answer that works by constraining the DTFT of $w[n]$ to be equal to the DTFT of $X(e^{j\omega})$ at 10 points in frequency, which would include the 5 required by the problem. Then we time-alias $x[n]$ to the $0 \leq n < 10$ range to obtain

$$w[n] = \left(\frac{1}{2}\right)^n \left(\frac{1}{1 - (1/2)^{10}}\right) \text{ for } 0 \leq n < 10$$

as a possible answer, which satisfies the constraint above.

Problem 6 (10%)

A system for the discrete-time spectral analysis of continuous-time signals is shown in Figure 6-1.

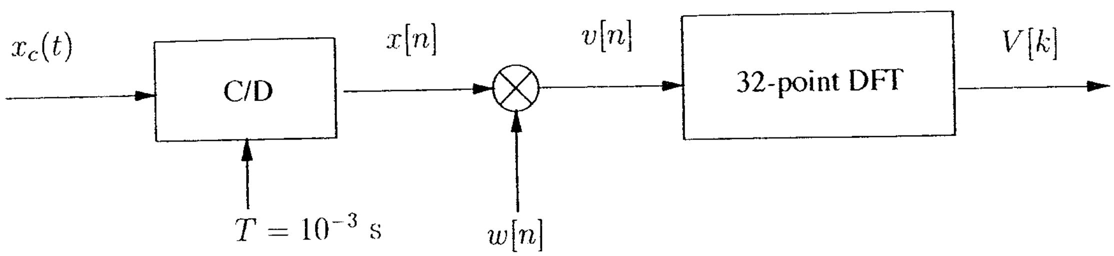


Figure 6-1: Spectral analysis system.

$w[n]$ is a rectangular window of length 32:

$$w[n] = \begin{cases} (1/32), & 0 \le n \le 31 \\ 0, & \text{otherwise} \end{cases}$$

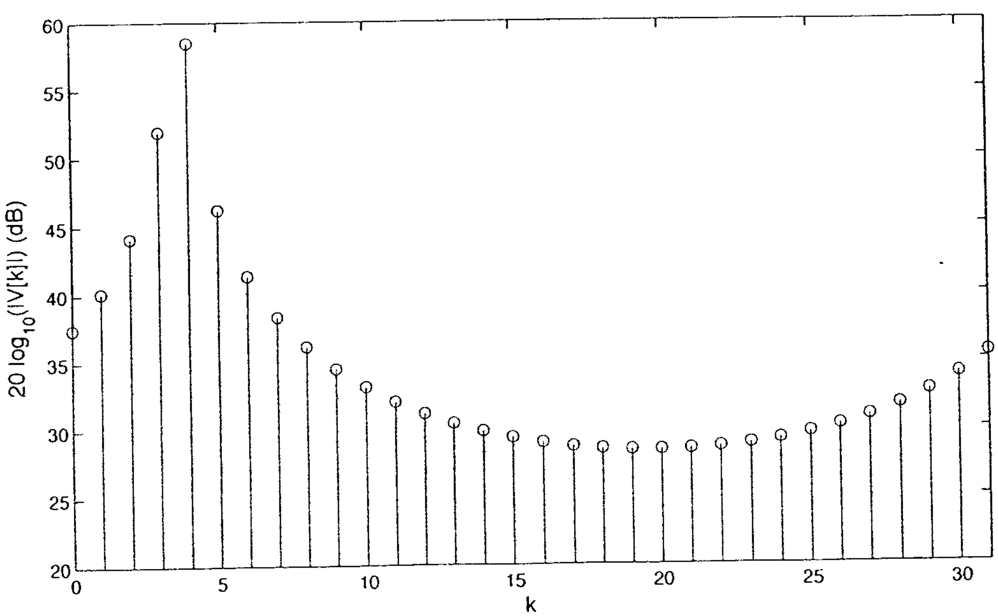


Figure 6-2: Output $|V[k]|$ in dB

Listed below are ten signals, at least one of which was the input $x_c(t)$. Indicate which signal(s) could have been the input $x_c(t)$ which produced the plot of $|V[k]|$ shown in dB units in Figure 6-2. As always, provide reasoning for your choice(s).

$$x_1(t) = 1000\cos(230\pi t) \qquad x_6(t) = 1000e^{j(250)\pi t}$$
$$x_2(t) = 1000\cos(115\pi t) \qquad x_7(t) = 10\cos(250\pi t)$$
$$x_3(t) = 10e^{j(460)\pi t} \qquad x_8(t) = 1000\cos(218.75\pi t)$$
$$x_4(t) = 1000e^{j(230)\pi t} \qquad x_9(t) = 10e^{j(200)\pi t}$$
$$x_5(t) = 10e^{j(230)\pi t} \qquad x_{10}(t) = 1000e^{j(187.5)\pi t}$$

Work to be looked at and answer:

- $x_c(t)$ cannot be a cosine, because a cosine would have a second peak in the negative frequencies, i.e. in the upper half of the DFT.
  $x_1(t)$, $x_2(t)$, $x_7(t)$, $x_8(t)$ are eliminated.
- $|V[4]| \approx 60\,dB \approx 1000$
  $x_3(t)$, $x_5(t)$, $x_9(t)$ are eliminated because their amplitudes are too low to have produced a peak magnitude of 1000 in the DFT
- The CT frequency of $x_c(t)$ cannot correspond exactly to a frequency $\omega_k = \frac{2\pi k}{32}$ sampled by the DFT. Otherwise, the DFT would be non-zero at exactly one value of k.
  $x_6(t)$ $(250\pi \rightarrow \frac{\pi}{4} \rightarrow k=4)$ and
  $x_{10}(t)$ $(187.5\pi \rightarrow \frac{3\pi}{16} \rightarrow k=3)$ are eliminated
- $\boxed{x_4(t)}$ is the only signal that could have been the input $x_c(t)$

Problem 7 (12%)

$x[n]$ is a finite-length sequence of length 1024, i.e.

$$x[n] = 0 \quad \text{for } n < 0,\ n > 1023.$$

The autocorrelation of $x[n]$ is defined as

$$R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m],$$

and $X_N[k]$ is defined as the N-point DFT of $x[n]$, with $N \geq 1024$.

We are interested in computing $R_{xx}[m]$. A proposed procedure begins by first generating the N-point inverse DFT of $|X_N[k]|^2$ to obtain an N-point sequence $g_N[n]$, i.e.

$$g_N[n] = \text{N-point IDFT}\left\{|X_N[k]|^2\right\}.$$

[6%] (a) Determine the minimum value of N so that $R_{xx}[m]$ can be obtained from $g_N[n]$. Also specify how you would obtain $R_{xx}[m]$ from $g_N[n]$.

Work to be looked at and answer:

$$R_{xx}[m] = x[n] * x[-n]\Big|_{n=m} = 0 \text{ for } m < -1023,\ m > 1023$$

$$|X_N[k]|^2 = X_N^*[k]X_N[k] = \text{N-point DFT}\left\{x[((-n))_N] \circledast x[n]\right\}$$

N-point circular convolution

$$g_N[m] = \text{N point IDFT}\left\{\text{N point DFT}\left\{x[((-n))_N] \circledast x[n]\right\}\right\}$$

$$g_N[m] = \begin{cases} x[((-n))_N] \circledast x[n]\Big|_{n=m} & \text{for } 0 \leq m \leq N-1 \\ 0 & \text{o.w.} \end{cases}$$

For $\boxed{N = 2047}$ and $0 \leq m \leq N-1$:

$$g_N[m] = x[((-n))_{2047}] * x[n]\Big|_{n=m}$$

For smaller N, circular convolution introduces time-aliasing.

To obtain $R_{xx}[m]$, use:

$$R_{xx}[m] = \begin{cases} g_{2047}[m] & \text{for } 0 \leq m < 1024 \\ g_{2047}[2047+m] & \text{for } -1024 < m \leq -1 \end{cases}$$

Name:________________________________

[6%] (b) Determine the minimum value of N so that $R_{xx}[m]$ for $|m| \leq 10$ can be obtained from $g_N[n]$. Also specify how you would obtain these values from $g_N[n]$.

Work to be looked at and answer:

For $0 \leq m \leq N-1$,

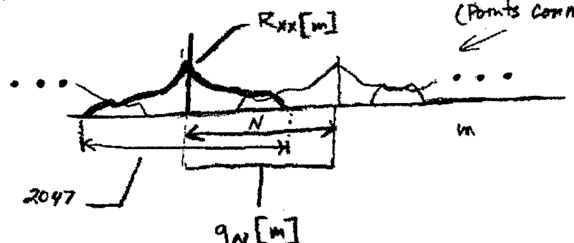
$$g_N[m] = \underbrace{x[((-n))_N] \circledast x[n]}_{N\text{-point circular convolution.}}\Big|_{n=m}$$

We'd now like to use a variant of the earlier technique but for smaller N. For general even N, our "post-processing" step is:

$$\hat{R}_{xx}[m] = \begin{cases} g_N[m] & \text{for } 0 \leq m < \frac{N+1}{2} \\ g_N[N+m] & \text{for } \frac{-(N+1)}{2} < m \leq -1 \end{cases}$$

If we want $\hat{R}_{xx}[m] = R_{xx}[m]$ for $|m| \leq 10$, we need to ensure that the time aliasing from circular convolution does not affect $g_N[m]$ for $0 \leq m \leq 10$ and for $N-11 \leq m \leq N-1$. ①

(Points connected for illustration)



For the lowest possible $N = 1024$, we have only $g_N[0]$ unaffected by aliasing. For $N = 1025$, $g[0]$, $g[1]$, and $g[1024]$ are unaffected, etc. Keeping this trend in mind and to satisfy ①, we pick

$$\boxed{N = 1034}$$

Our post-processing step becomes:

$$R_{xx}[m] = \begin{cases} g_{1034}[m] & \text{for } 0 \leq m \leq 10 \\ g_{1034}[1034+m] & \text{for } -10 \leq m \leq -1 \end{cases}$$

Problem 8 (8%)

A system for examining the spectral content of a signal $x[n]$ is shown in Figure 8-1. The filters $h[n]$ in each channel are identical three-point non-causal FIR filters and have impulse response

$$h[n] = h_0\delta[n] + h_1\delta[n+1] + h_2\delta[n+2].$$

The filter outputs are sampled at $n = 0$ to obtain the sequence $y_k[0]$, $k = 0, 1, 2, 3$.

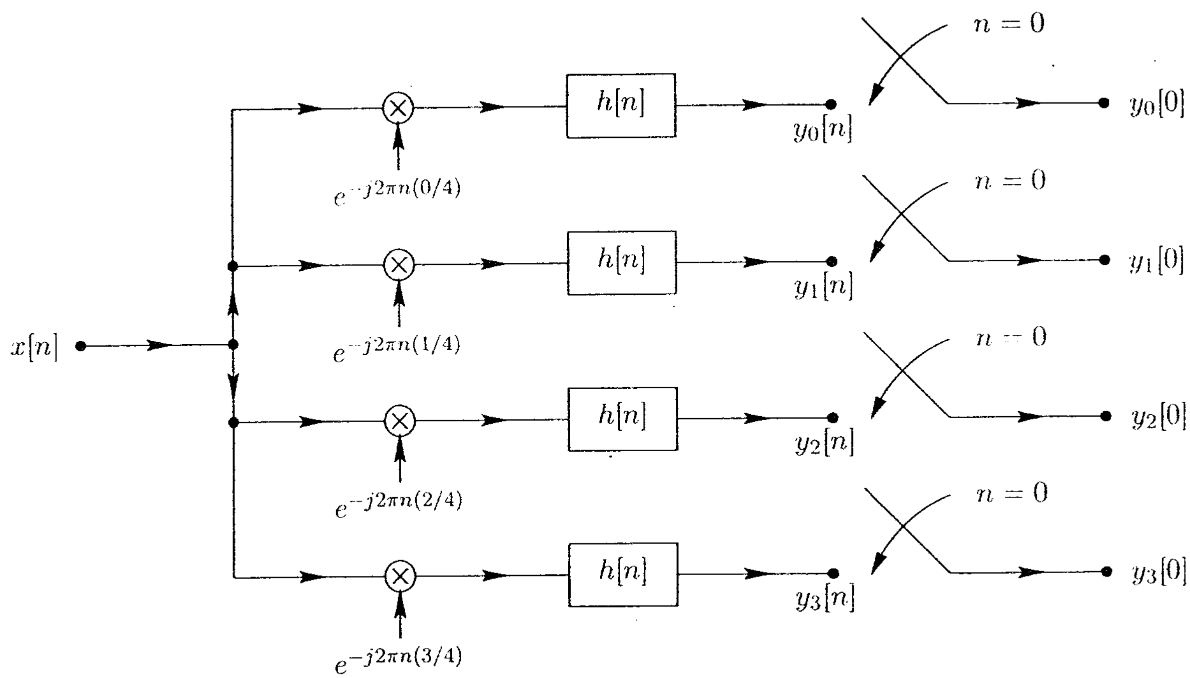


Figure 8-1: Filter bank network.

An alternative to the system in Figure 8-1 has been proposed using a 4-point DFT as shown in Figure 8-2.

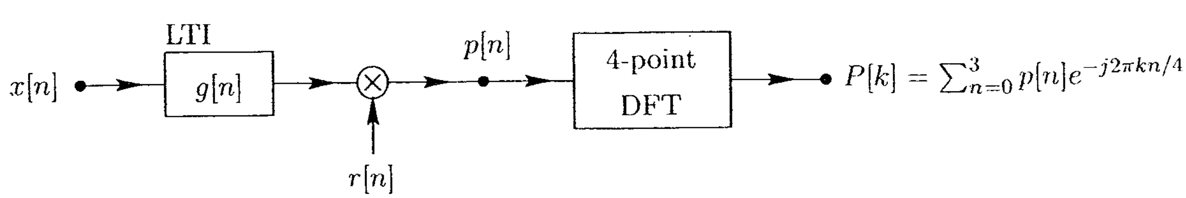


Figure 8-2: Alternative system.

Name:____

Determine $g[n]$ and $r[n]$ so that $P[k] = y_k[0]$.

Work to be looked at and answer:

For Figure 8-1,

$$y_K[0] = \sum_{n=-\infty}^{\infty} h[0-n]\, x[n]\, e^{-j2\pi nk/4}$$

Since $h[n]$ is non-zero only for $n = 0, -1, -2$, the summation limits become $n = 0$ and $n = 2$:

$$y_K[0] = \sum_{n=0}^{2} h[-n]\, x[n]\, e^{-j2\pi nk/4}$$

For Figure 8-2, let $g[n] = \delta[n]$ be an identity system for now.

$$P[K] = \sum_{n=0}^{3} p[n]\, e^{-j2\pi Kn/4}$$

$$= \sum_{n=0}^{3} r[n]\, x[n]\, e^{-j2\pi Kn/4}$$

We can make $P[K] = y_K[0]$ by letting $r[n] = h[-n]$,

$$\boxed{r[n] = h_0\,\delta[n] + h_1\,\delta[n-1] + h_2\,\delta[n-2]}$$

so that $r[3] = 0$, and keep

$$\boxed{g[n] = \delta[n]}$$

Problem 9 (9%)

Consider the system shown in Figure 9-1. The subsystem from $x[n]$ to $y[n]$ is a causal, LTI system implementing the difference equation

$$y[n] = x[n] + ay[n-1].$$

$x[n]$ is a finite length sequence of length 90, i.e.

$$x[n] = 0 \quad \text{for } n < 0 \text{ and } n > 89.$$

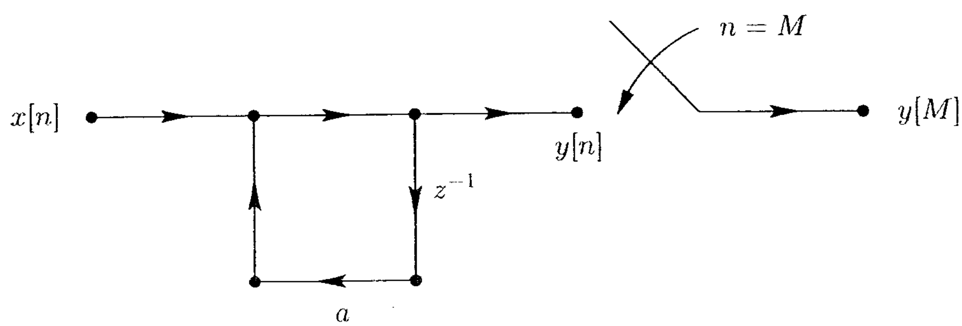


Figure 9-1: System for calculating $y[M]$ from $x[n]$.

Determine a choice for the complex constant a and a choice for the sampling instant M so that

$$y[M] = X(e^{j\omega})\big|_{\omega=2\pi/60}$$

Work to be looked at and answer:

To compute the DTFT of $x[n]$ at a particular frequency point we need the impulse response of the LTI filter to be a complex exponential. If $a = W_N^{-k} = e^{j\frac{2\pi k}{N}}$, we can write

$$y[M] = \sum_{n=0}^{M} x[n] W_N^{-k(M-n)}$$

We need the output sample to be equal to the DTFT of $x[n]$ evaluated at $\omega = \frac{2\pi}{60}$, *i.e.*

$$\sum_{n=0}^{M} x[n] W_N^{nk} W_N^{-Mk} = \sum_{n=0}^{89} x[n] W_{60}^{n}$$

We can see that we need $M \geq 89$, otherwise samples of $x[n]$ will be disregarded in the computation. If M is chosen to be an integer multiple of N, then $W_N^{-Mk} = 1$ and we eliminate that term. All that remains is to choose k and N such that $\frac{k}{N} = \frac{1}{60}$.

If $k = 1$, $N = 60$ and $M = 120$ we have

$$y[M] = \sum_{n=0}^{120} x[n] W_{60}^{n} W_{60}^{-120} = \sum_{n=0}^{89} x[n] W_{60}^{n}$$

So we can use $a = W_{60}^{-1} = e^{j\frac{2\pi}{60}}$ and $M = 120$

In fact, any M that is a multiple of 60 and is greater than 89 will work with this choice of a, due to the periodicity of W_{60}^{n}.