Introduction to Simulation - Lecture 8

1-D Nonlinear Solution Methods

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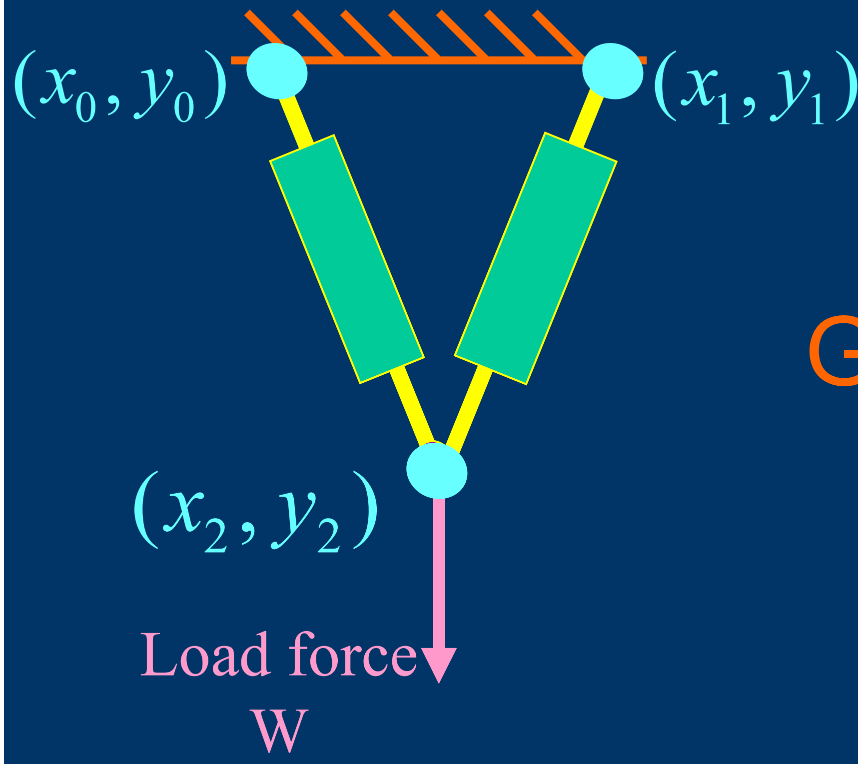
Thanks to Deepak Ramaswamy Jaime Peraire, Michal Rewienski, and Karen Veroy

Outline

- Nonlinear Problems
  - Struts and Circuit Example
- Richardson and Linear Convergence
  - Simple Linear Example
- Newton's Method
  - Derivation of Newton
  - Quadratic Convergence
  - Examples
  - Global Convergence
  - Convergence Checks

Nonlinear problems

Strut Example



Given: x0, y0, x1, y1, W

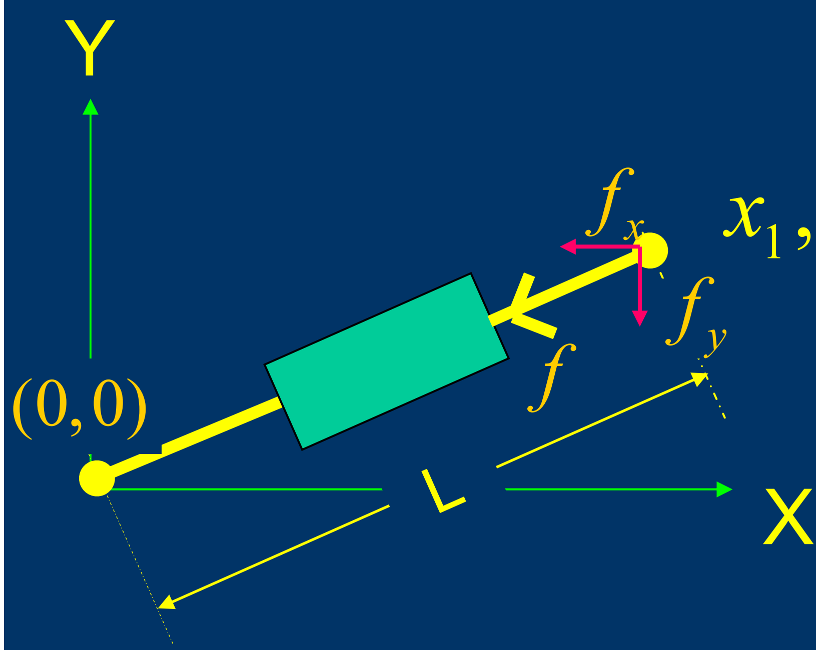
Find: x2, y2

Need to Solve $\sum f_x = 0$ $\sum f_y + W = 0$

Nonlinear Problems

Struts Example

Reminder: Strut Forces



$$f = EA_c \frac{L_0 - L}{L_0} = \varepsilon\left(L_0 - L\right)$$

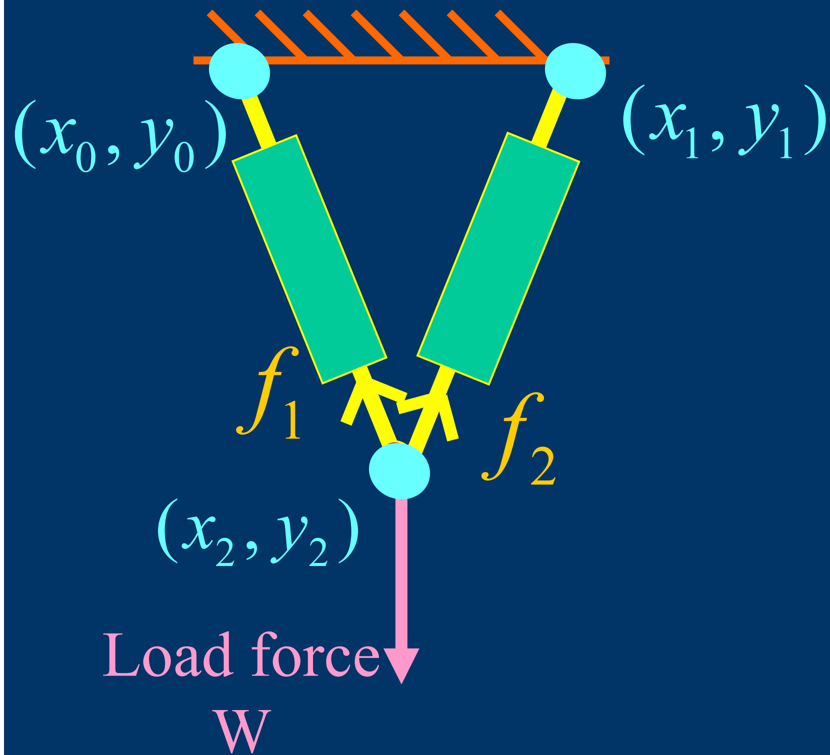
$$f_x = \frac{x_1}{L} f$$

$$f_y = \frac{y_1}{L} f$$

$$L = \sqrt{x_1^{\,2} + y_1^{\,2}}$$

Nonlinear problems

Strut Example



$$L_1 = \sqrt{\left(x_2 - x_0\right)^2 + \left(y_2 - y_0\right)^2}$$

$$L_2 = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$f_{1x} = \frac{x_2 - x_0}{L_1}\varepsilon(L_o - L_1)$$

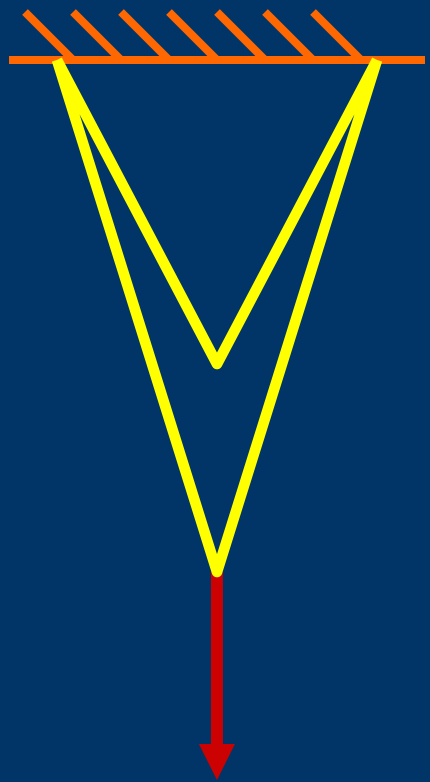
$$f_{2x} = \frac{x_2 - x_1}{L_2}\varepsilon(L_o - L_2)$$

$$\sum f_{1x} + f_{2x} = 0 \qquad \sum f_{1y} + f_{2y} + W = 0$$

Nonlinear problems

Strut Example

Why Nonlinear?



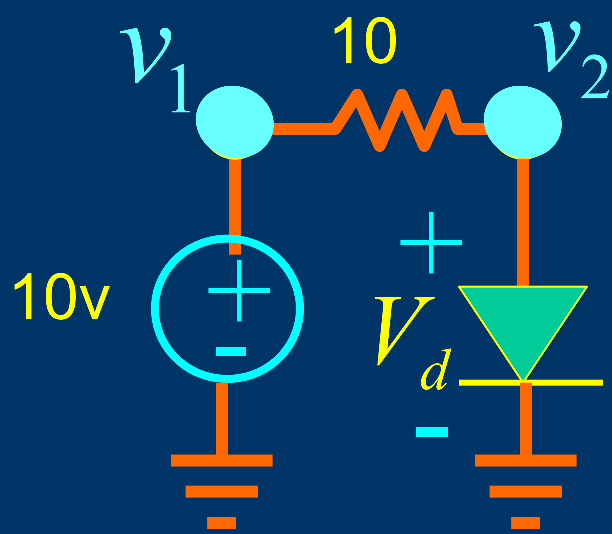
$$\frac{y_2 - y_1}{L_2}\varepsilon(L_o - L_2) +$$

$$\frac{y_2 - y_0}{L_1}\varepsilon(L_o - L_1) + W = 0$$

Pull Hard on the Struts → The strut forces change in both magnitude and direction

Nonlinear problems

Circuit Example



$$I_r - \frac{1}{10} V_r = 0$$

$$I_d - I_s\left(e^{V_d / V_t} - 1\right) = 0$$

Need to Solve

$$I_d + I_r = 0$$

$$I_{v_{src}} - I_r = 0$$

Nonlinear problems

Solve Iteratively

Hard to find analytical solution for $f(x)=0$

Solve iteratively

guess at a solution $x^0 = x_0$

repeat for k = 0, 1, 2, ….

$$x^{k+1} = W\left(x^k\right)$$

until $f\left(x^{k+1}\right) \approx 0$

Ask

- Does the iteration converge to correct solution ?
- How fast does the iteration converge?

Richardson Iteration

Definition

Richardson Iteration Definition

$$x^{k+1} = x^k + f(x^k)$$

An iteration stationary point is a solution

$$x^{k+1} = x^k$$

$$\Rightarrow \; f(x^k) = 0$$

$$\Rightarrow x^k = x^* \; (Solution)$$

Richardson Iteration

Example 1

$$f(x) = -0.7x + 10$$

Start with $x^0 = 0$

$$x^1 = x^0 + f(x^0) = 10$$

$$x^2 = x^1 + f(x^1) = 13$$

$$x_3 = x^2 + f(x^2) = 13.9$$

$$x_4 = x^3 + f(x^3) = 14.17$$

$$x^5 = 14.25$$

$$x^6 = 14.27$$

$$x^7 = 14.28$$

$$x^8 = 14.28$$

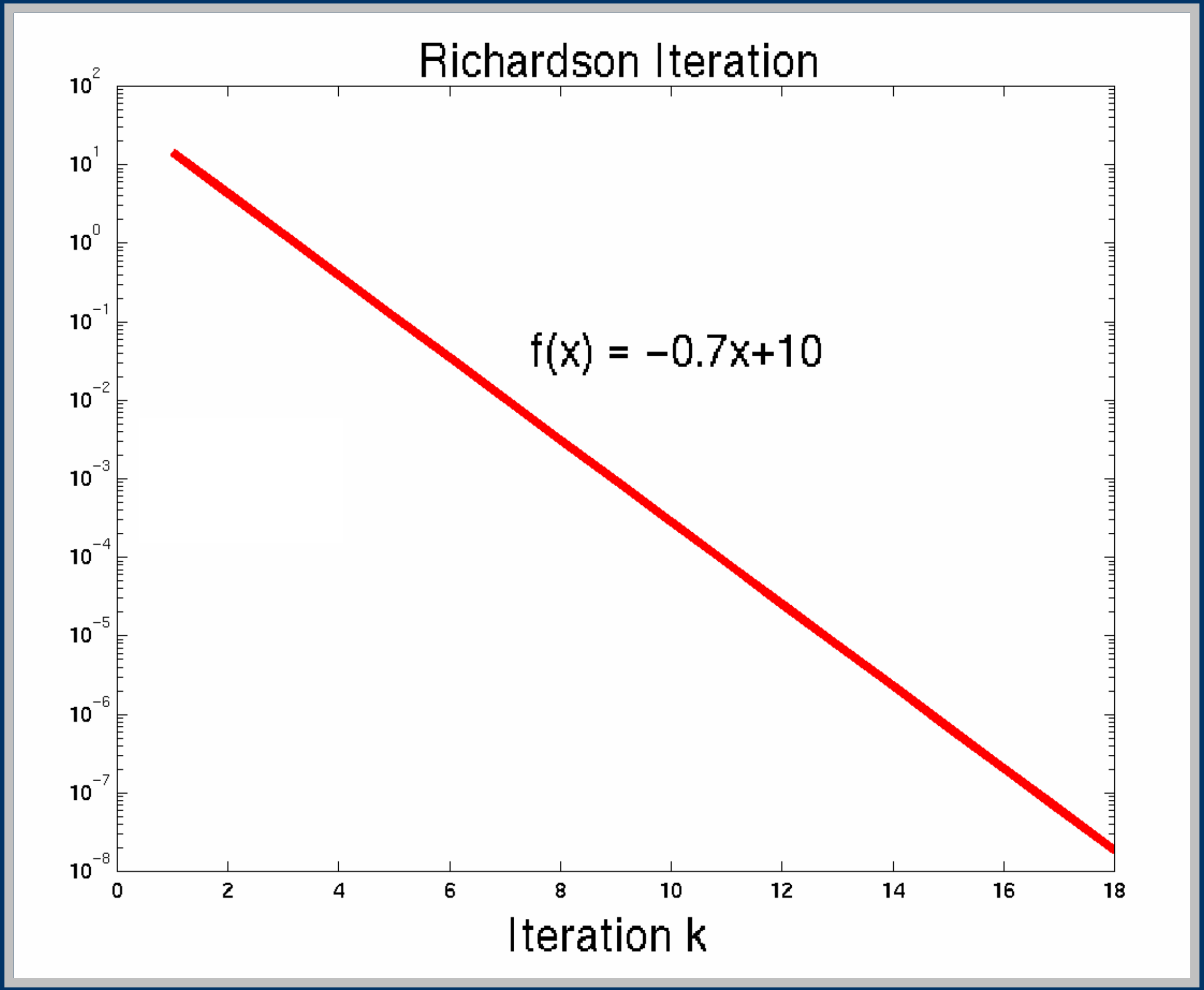
Converged

Richardson Iteration

Example 1

$$f(x) = -0.7x + 10$$

$$\left| x^k - x^* \right|$$



Richardson Iteration

Example 2

$$f(x) = 2x + 10$$

Start with $x_0 = 0$

$$x_1 = x_0 + f(x_0) = 10$$

$$x_2 = x_1 + f(x_1) = 40$$

$$x_3 = x_2 + f(x_2) = 130$$

$$x_4 = x_3 + f(x_3) = 400$$

No convergence !

Richardson Iteration

Convergence

Setup

Iteration Equation $x^{k+1} = x^k + f(x^k)$

Exact Solution $x^* = x^* + \underbrace{f(x^*)}_{=0}$

Computing Differences

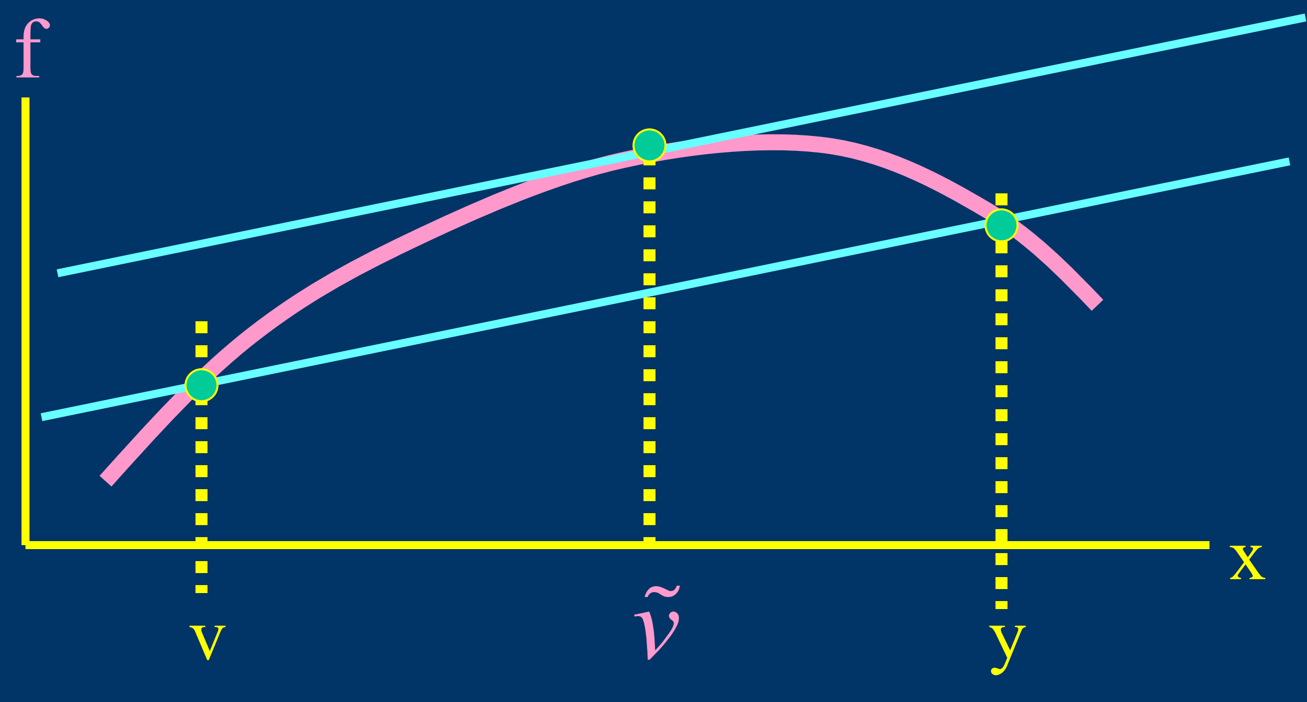
$$x^{k+1} - x^* = x^k - x^* + \underbrace{f(x^k) - f(x^*)}_{\text{Need to Estimate}}$$

Richardson Iteration

Convergence

Mean Value Theorem

$$f(v) - f(y) = \frac{\partial f(\tilde{v})}{\partial x}(v - y) \qquad \tilde{v} \in [v, y]$$



Richardson Iteration

Convergence

Use MVT

Iteration Equation $$x^{k+1} = x^k + f(x^k)$$

Exact Solution $$x^* = x^* + \underbrace{f(x^*)}_{=0}$$

Computing Differences

$$x^{k+1} - x^* = x^k - x^* + f(x^k) - f(x^*)$$

$$= \left(1 + \frac{\partial f(\tilde{x})}{\partial x}\right)\left(x^k - x^*\right)$$

Richardson Iteration

Convergence

Richardson Theorem

If $\left|1+\frac{\partial f(\tilde{x})}{\partial x}\right| \le \gamma < 1$ for all $\tilde{x}$ s.t. $\left|\tilde{x}-x^*\right| < \delta$

And $\left|x^0 - x^*\right| < \delta$

Then $\left|x^{k+1} - x^*\right| \le \gamma \left|x^k - x^*\right|$

Or $\lim_{k\to\infty} \left|x^{k+1} - x^*\right| = \lim_{k\to\infty} \gamma^k \left|x^0 - x^*\right| = 0$

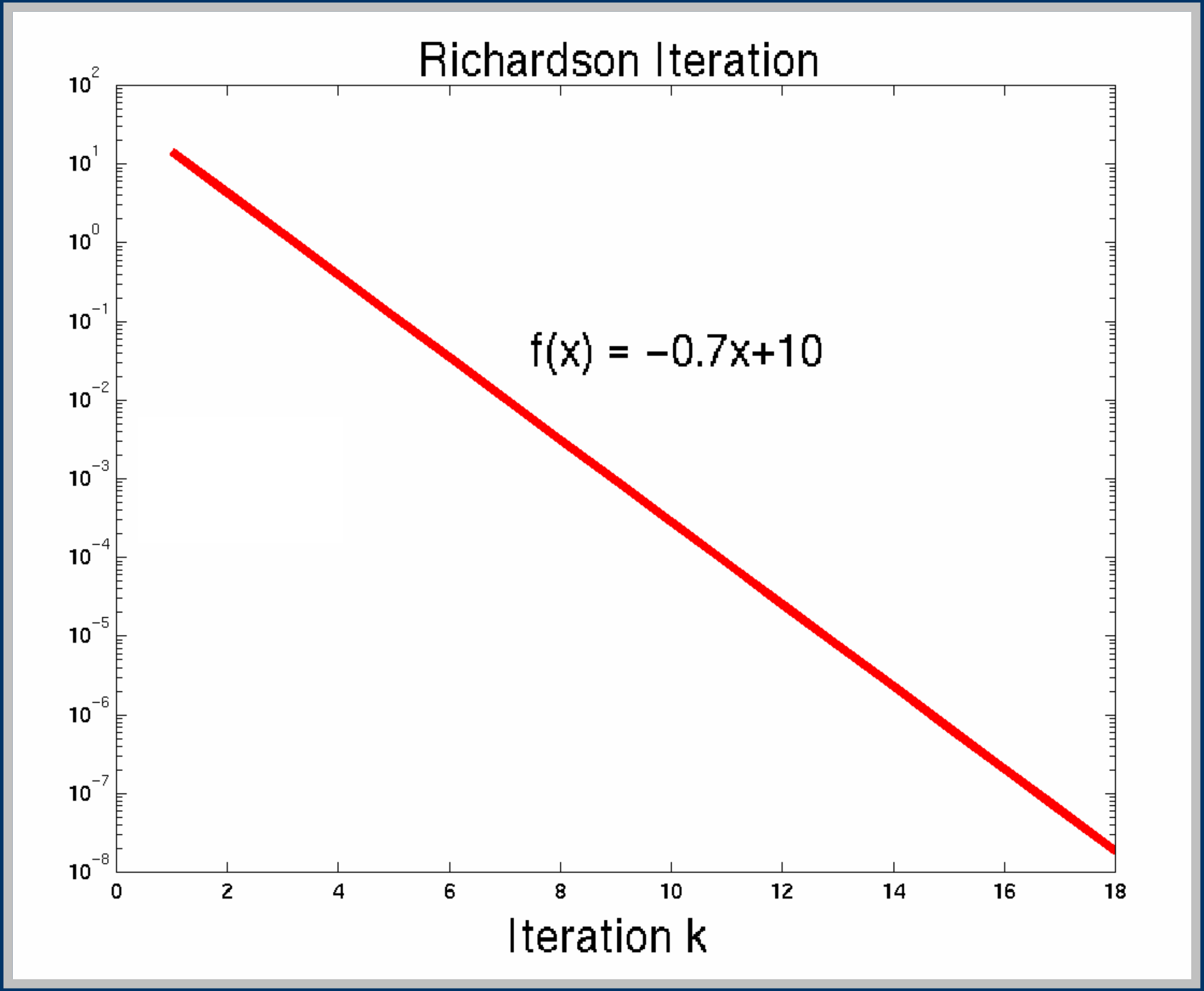
Linear Convergence

Richardson Iteration

Example 1

$$f(x) = -0.7x + 10$$

$$\left| x^k - x^* \right|$$



Richardson Iteration

Problems

- Convergence is only linear
- *x, f(x)* not in the same units:
  - *x* is a voltage, *f(x)* a current in circuits
  - *x* is a displacement, *f(x)* a force in struts
  - Adding 2 different physical quantities
- But a Simple Algorithm
  - Just calculate *f(x)* and update

Another approach

Newton's method

From the Taylor series about solution

$$0 = f(x^*) \simeq f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k)$$

Define iteration

Do $k = 0$ to ….

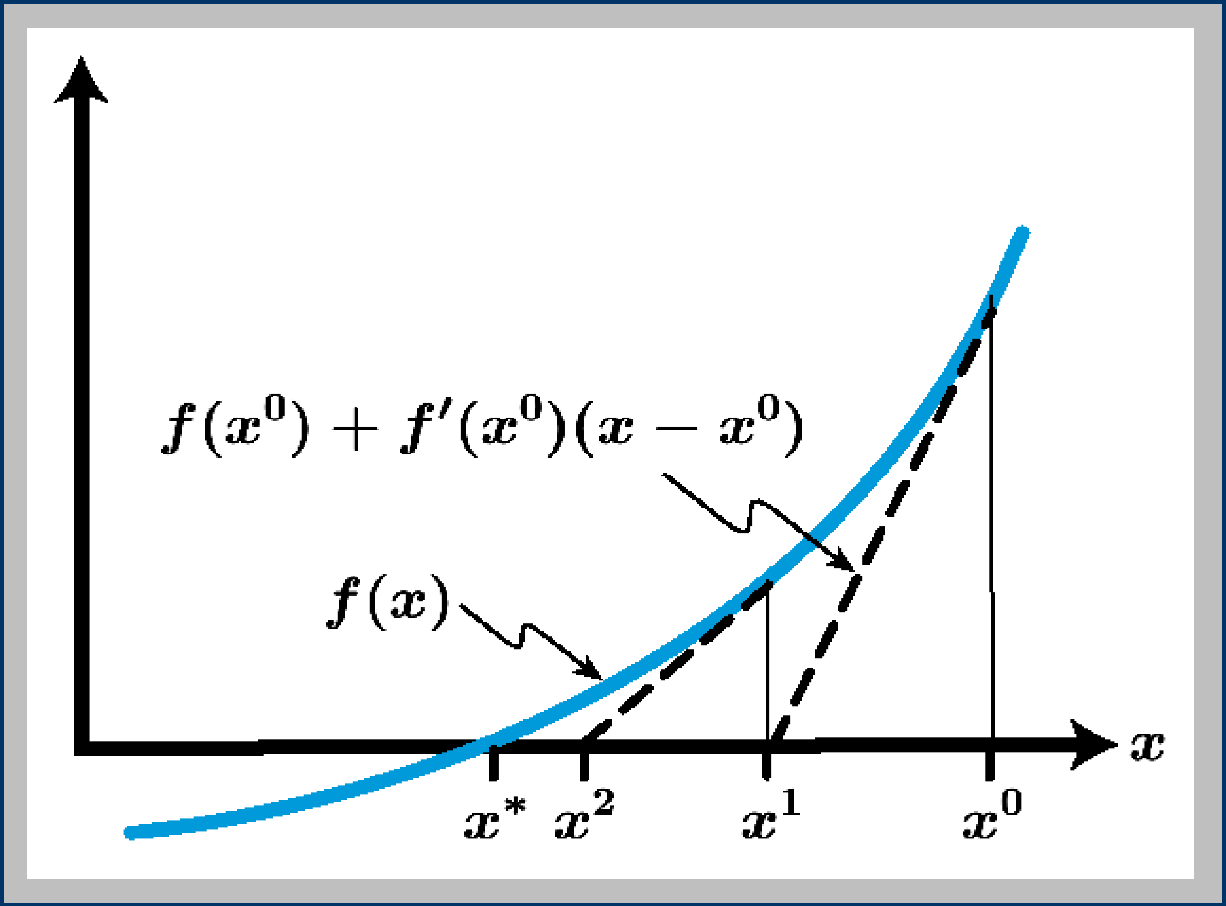
$$x^{k+1} = x^k - \left[\frac{df}{dx}(x^k)\right]^{-1} f(x^k)$$

$$if \quad \left[\frac{df}{dx}(x^k)\right]^{-1} \quad exists$$

until convergence

Newton's Method

Graphically



Newton's Method

Example

EXAMPLE : $f(x) = x^3 - 2, \quad x^* = \sqrt[3]{2} \approx 1.259921$

k	x^k	$\lvert x^k - x^* \rvert$
0	10.0	8.740
1	6.673333	5.413
.	.	.
8	1.261665	$1.744e-03$
9	1.259924	$2.410e-06$
10	1.259921	$4.609e-12$

Asymptotically,

$$|x^{k+1} - x^*| \approx C|x^k - x^*|^\alpha$$

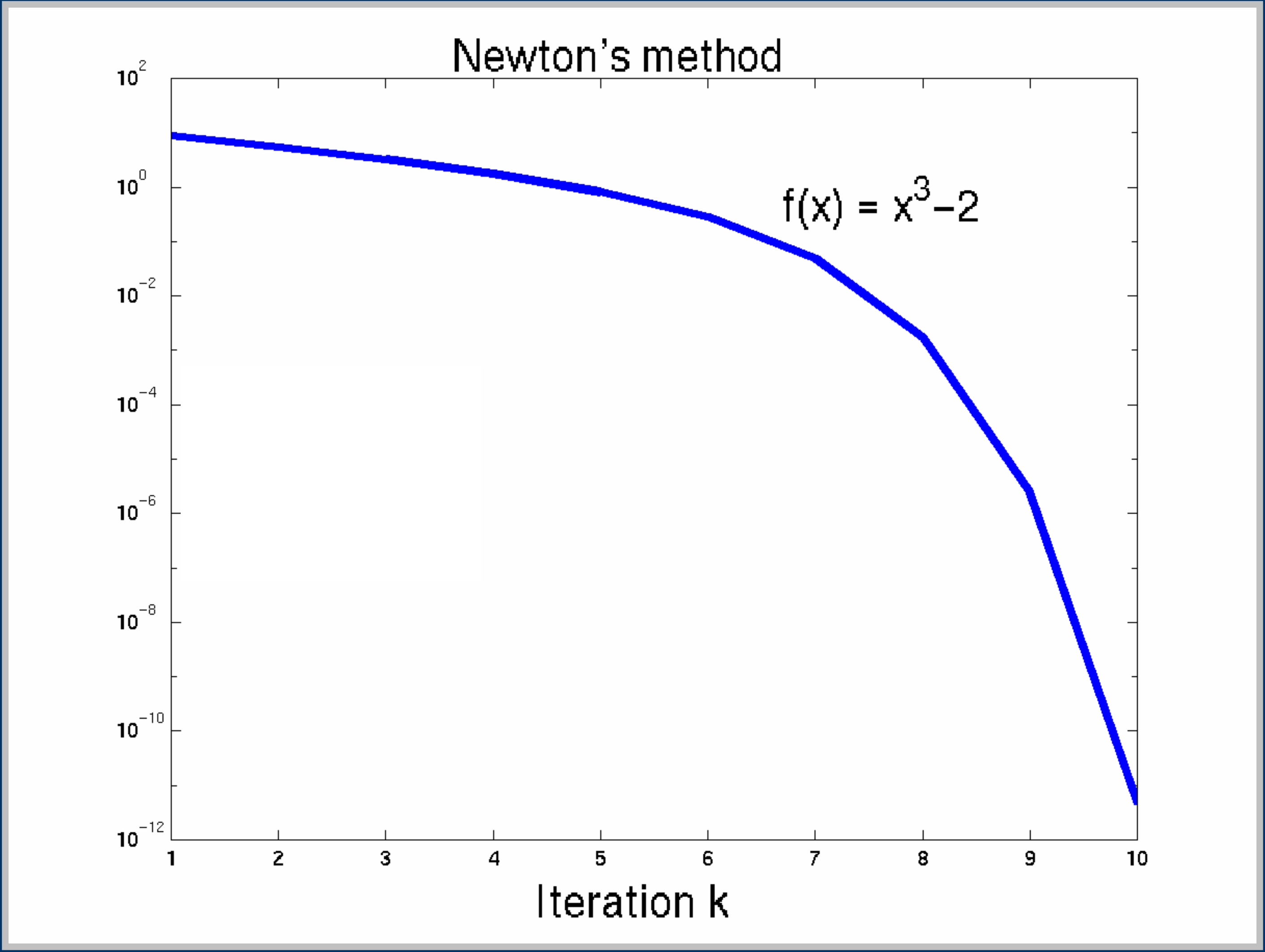
$C = 0.7951$

$\alpha = 2.000$ Quadratic

Newton's Method

Example

$$\left| x^k - x^* \right|$$



Newton's Method

Convergence

$$0 = f(x^*) = f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k) + \frac{d^2 f}{dx^2}(\tilde{x})(x^* - x^k)^2$$

some $\tilde{x} \in [x^k, x^*]$

Mean Value theorem truncates Taylor series

But

$$0 = f(x^k) + \frac{df}{dx}(x^k)(x^{k+1} - x^k)$$

by Newton definition

Newton's Method

Convergence

Contd.

Subtracting $$\frac{df}{dx}(x^k)(x^{k+1}-x^*) = \frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2$$

Dividing through $$(x^{k+1}-x^*) = [\frac{df}{dx}(x^k)]^{-1}\frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2$$

Suppose $$\left|\left[\frac{df}{dx}(x)\right]^{-1}\frac{d^2 f}{d^2 x}(x)\right| \le L \quad \text{for all } x$$

then $$\left|x^{k+1}-x^*\right| \le L\left|x^k - x^*\right|^2$$

Convergence is quadratic if L is bounded

Newton's Method

Convergence

Example 1

$$f(x) = x^2 - 1 = 0, \quad \textit{find} \;\; x \;\; (x^* = 1)$$

$$\frac{df}{dx}(x^k) = 2\,x^k$$

$$2x^k (x^{k+1} - x^k) = -\left(\left(x^k\right)^2 - 1\right)$$

$$2x^k (x^{k+1} - x^*) + 2x^k (x^* - x^k) = -\left(\left(x^k\right)^2 - \left(x^*\right)^2\right)$$

$$or \;\; (x^{k+1} - x^*) = \frac{1}{2x^k}(x^k - x^*)^2$$

Convergence is quadratic

Newton's Method

Convergence

Example 2

$$f(x) = x^2 = 0, \quad x^* = 0$$

$$\frac{df}{dx}(x^k) = 2\,x^k$$

Note : $\left(\frac{df}{dx}\right)^{-1}$ not bounded away from zero

$$\Rightarrow 2x^k(x^{k+1} - 0) = (x^k - 0)^2$$

$$x^{k+1} - 0 = \frac{1}{2}\left(x^k - 0\right) \qquad \text{for } x^k \neq x^* = 0$$

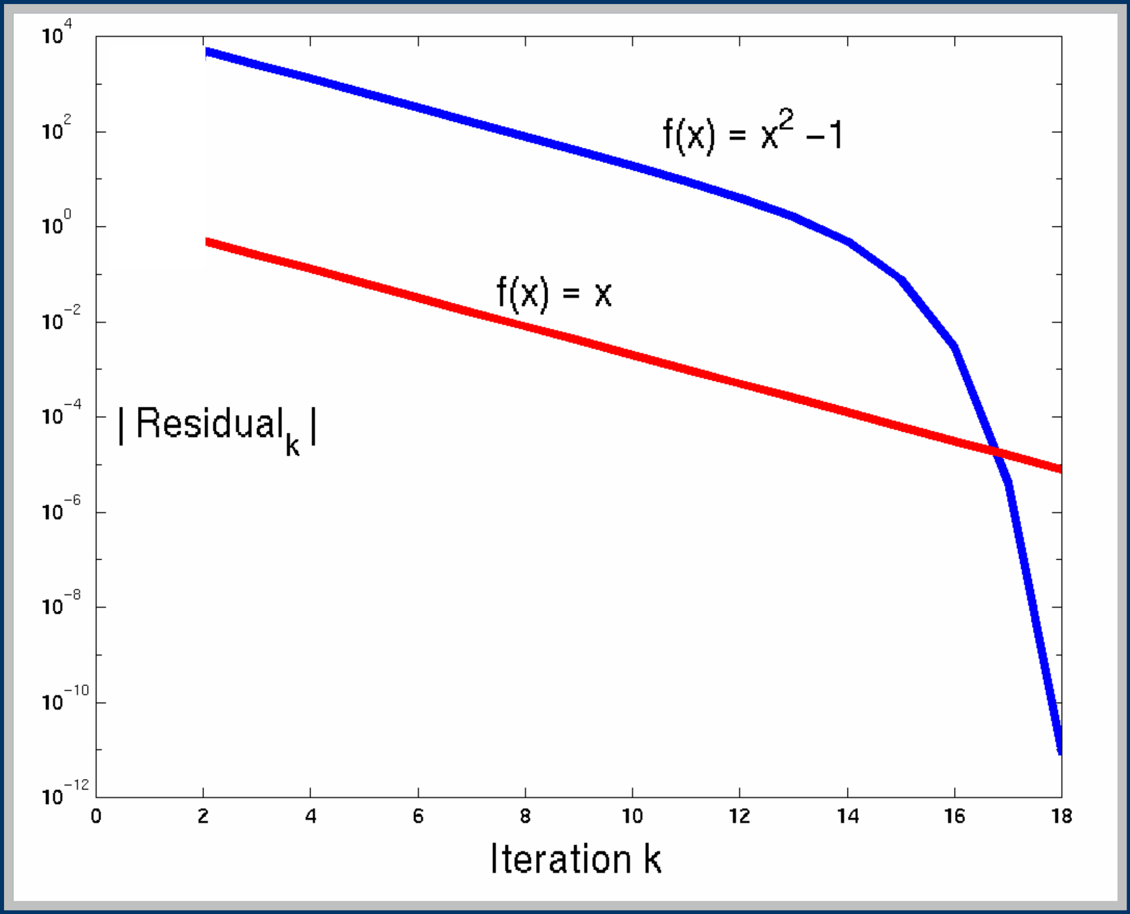
$$or\ (x_{k+1} - x^*) = \frac{1}{2}(x_k - x^*)$$

Convergence is linear

Newton's Method

Convergence

Examples 1 , 2



Newton's Method

Convergence

Suppose $\left| \left[\frac{df}{dx}(x) \right]^{-1} \frac{d^2 f}{d^2 x}(x) \right| \leq L$ for all x

if $L\left|x_0 - x^*\right| \leq \gamma < 1$

then x_k converges to x^*

Proof

$$\left|x_1 - x^*\right| \leq L\left|(x_0 - x^*)\right|\left|x_0 - x^*\right|$$

$$\Rightarrow \left|x_1 - x^*\right| \leq \gamma \left|x_0 - x^*\right|$$

$$\Rightarrow \left|x_2 - x^*\right| \leq L\gamma \left|x_0 - x^*\right|\left|x_1 - x^*\right|$$

$$or \left|x_2 - x^*\right| \leq \gamma^2 \left|x_1 - x^*\right| \leq \gamma^3 \left|x_0 - x^*\right|$$

$$\Rightarrow \left|x_3 - x^*\right| \leq \gamma^4 \left|x_2 - x^*\right| \leq \gamma^7 \left|x_0 - x^*\right|$$

Newton's Method

Convergence

Theorem

If L is bounded ($\frac{df}{dx}$ bounded away from zero ; $\frac{d^2 f}{dx^2}$ bounded) then Newton's method is guaranteed to converge given a "close enough" guess

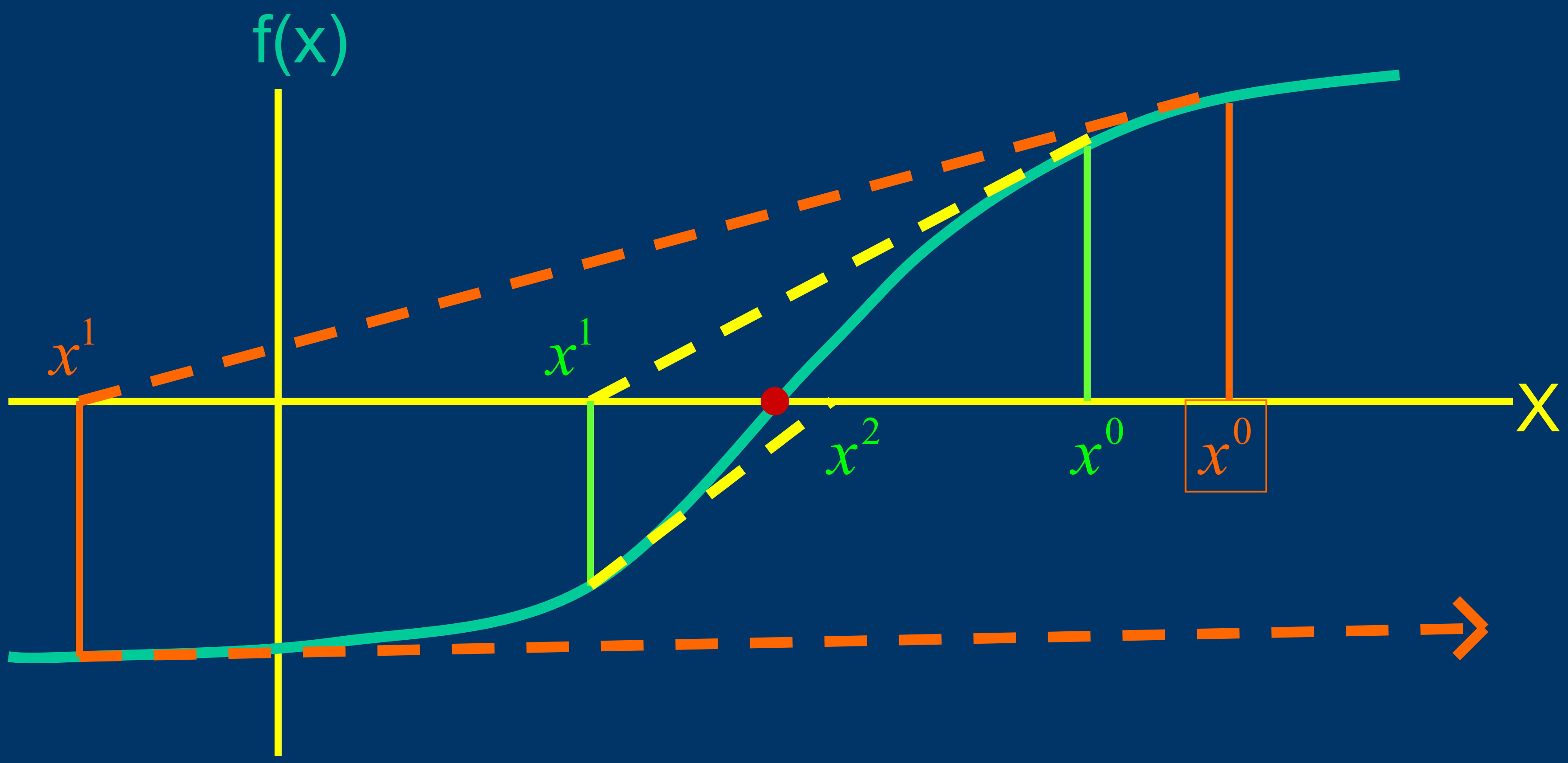
Always converges ?

Newton's Method

Convergence

Example

Convergence Depends on a Good Initial Guess

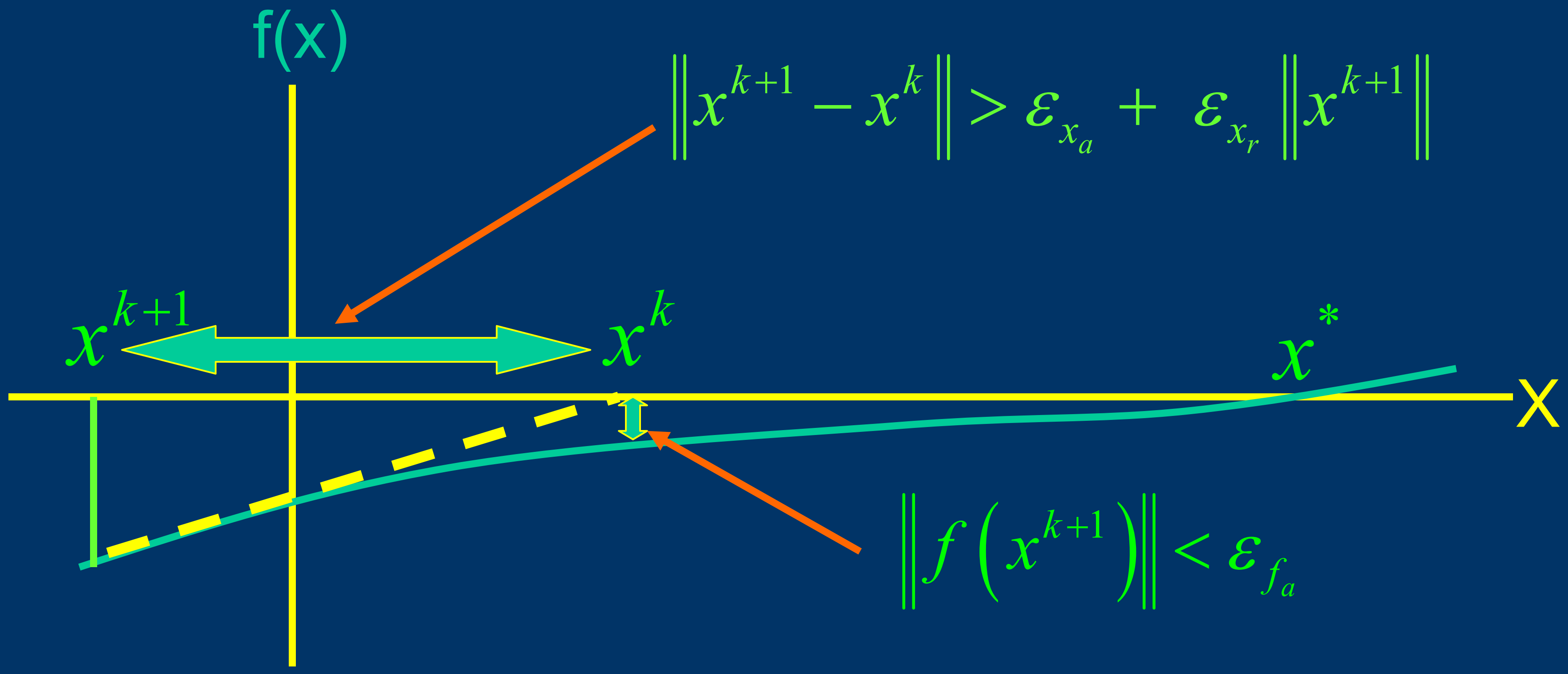


Newton's Method

Convergence

Convergence Checks

Need a "delta-x" check to avoid false convergence

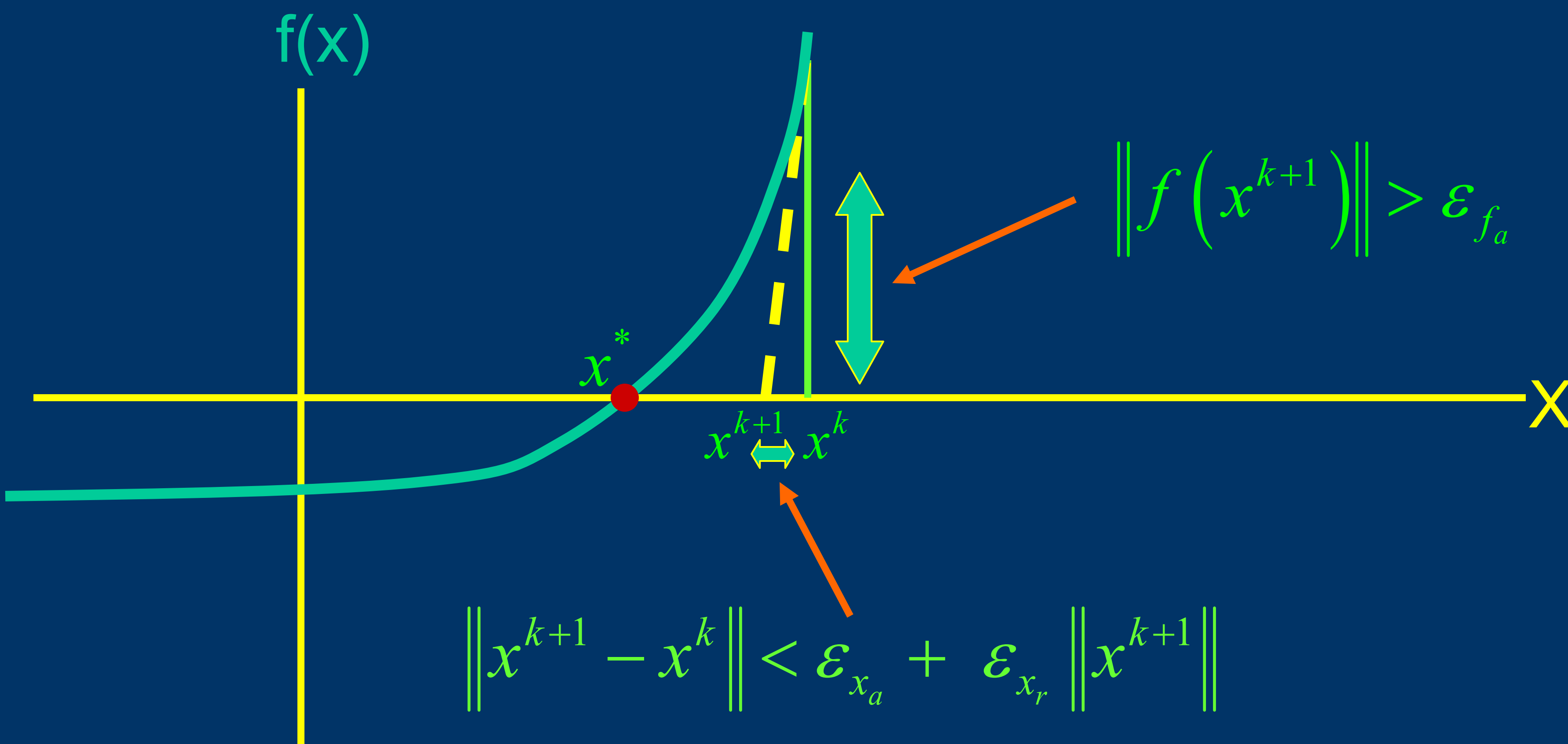


Newton's Method

Convergence

Convergence Checks

Also need an "$f(x)$" check to avoid false convergence



Summary

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  - Struts and Circuit Example
- Richardson and Linear Convergence
  - Simple Linear Example
- 1-D Newton's Method
  - Derivation of Newton
  - Quadratic Convergence
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