Introduction to Simulation - Lecture 19

Laplace's Equation – FEM Methods

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Thanks to Deepak Ramaswamy, Michal Rewienski, and Karen Veroy, Jaime Peraire and Tony Patera

Outline for Poisson Equation Section

- Why Study Poisson's equation
  - Heat Flow, Potential Flow, Electrostatics
  - Raises many issues common to solving PDEs.
- Basic Numerical Techniques
  - basis functions (FEM) and finite-differences
  - Integral equation methods
- Fast Methods for 3-D
  - Preconditioners for FEM and Finite-differences
  - Fast multipole techniques for integral equations

Outline for Today

- Why Poisson Equation
  - Reminder about heat conducting bar
- Finite-Difference And Basis function methods
  - Key question of convergence
- Convergence of Finite-Element methods
  - Key idea: solve Poisson by minimization
  - Demonstrate optimality in a carefully chosen norm

Drag Force Analysis of Aircraft

- Potential Flow Equations
  - Poisson Partial Differential Equations.

Engine Thermal Analysis

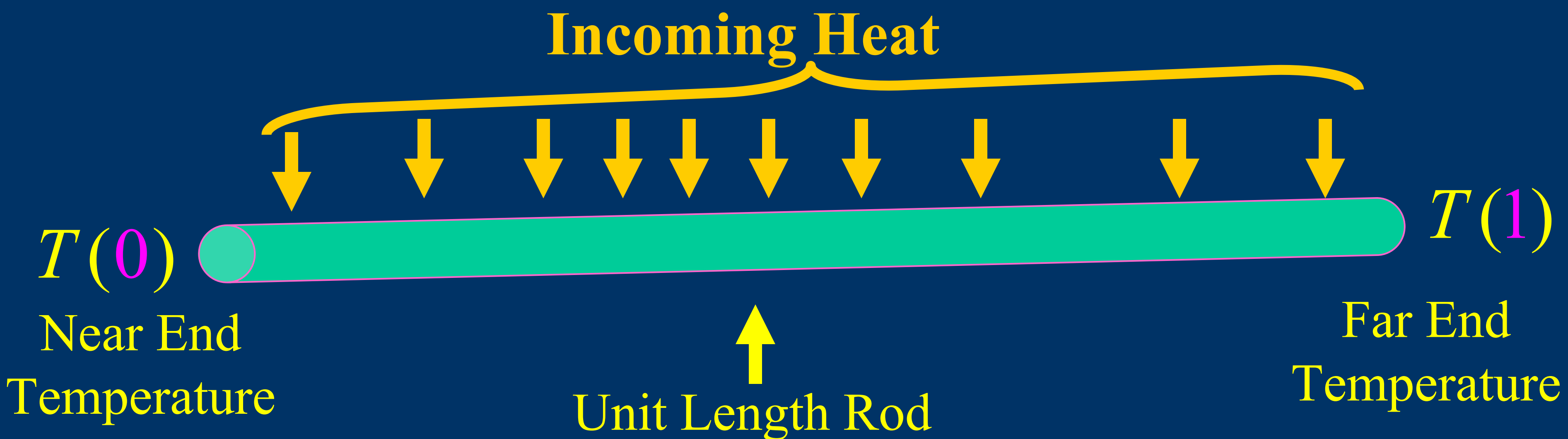
- Thermal Conduction Equations
  - The Poisson Partial Differential Equation.

Capacitance on a microprocessor Signal Line

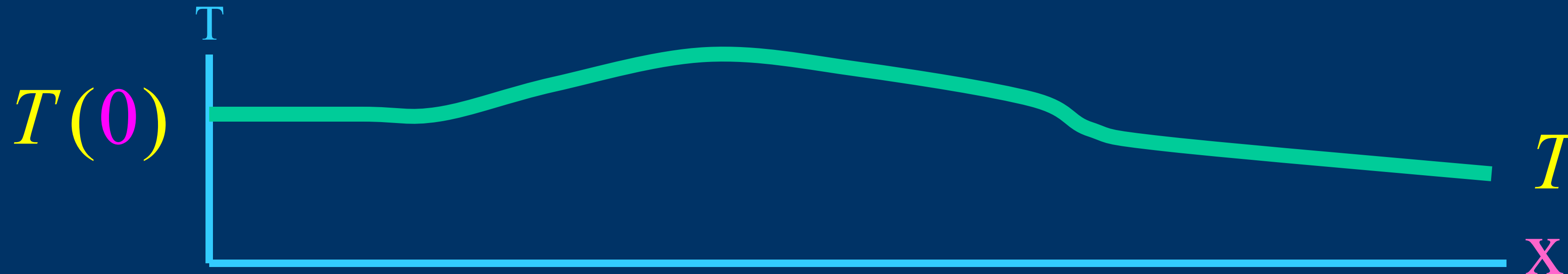
- Electrostatic Analysis
  - The Laplace Partial Differential Equation.

Heat Flow

1-D Example



Question: What is the temperature distribution along the bar



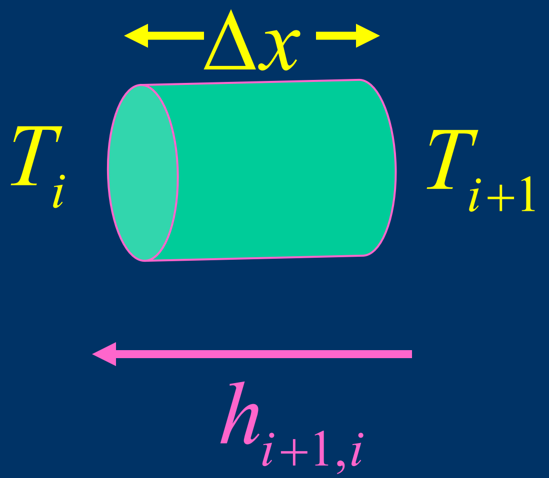
1-D Example

Discrete Representation

1) Cut the bar into short sections

2) Assign each cut a temperature

Heat Flow through one section

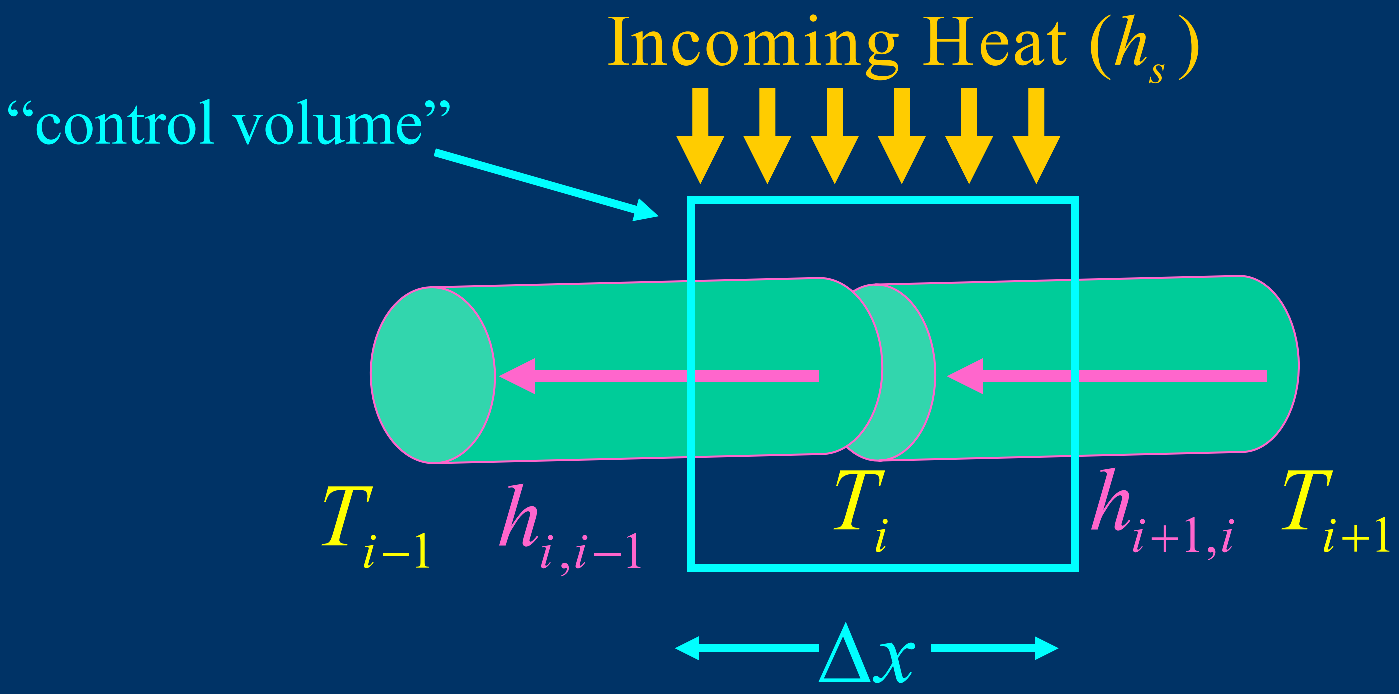


$$h_{i+1,i} = \text{ heat flow } = \kappa \frac{T_{i+1} - T_i}{\Delta x}$$

Limit as the sections become vanishingly small

$$\lim_{\Delta x \to 0} h(x) = \kappa \frac{\partial T(x)}{\partial x}$$

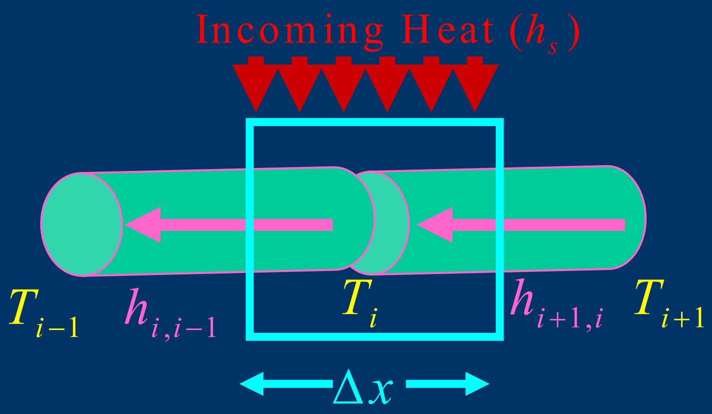
Two Adjacent Sections



Heat Flows into Control Volume Sums to zero

$$h_{i+1,i} - h_{i,i-1} = -h_s \Delta x$$

Heat Flows into Control Volume Sums to zero



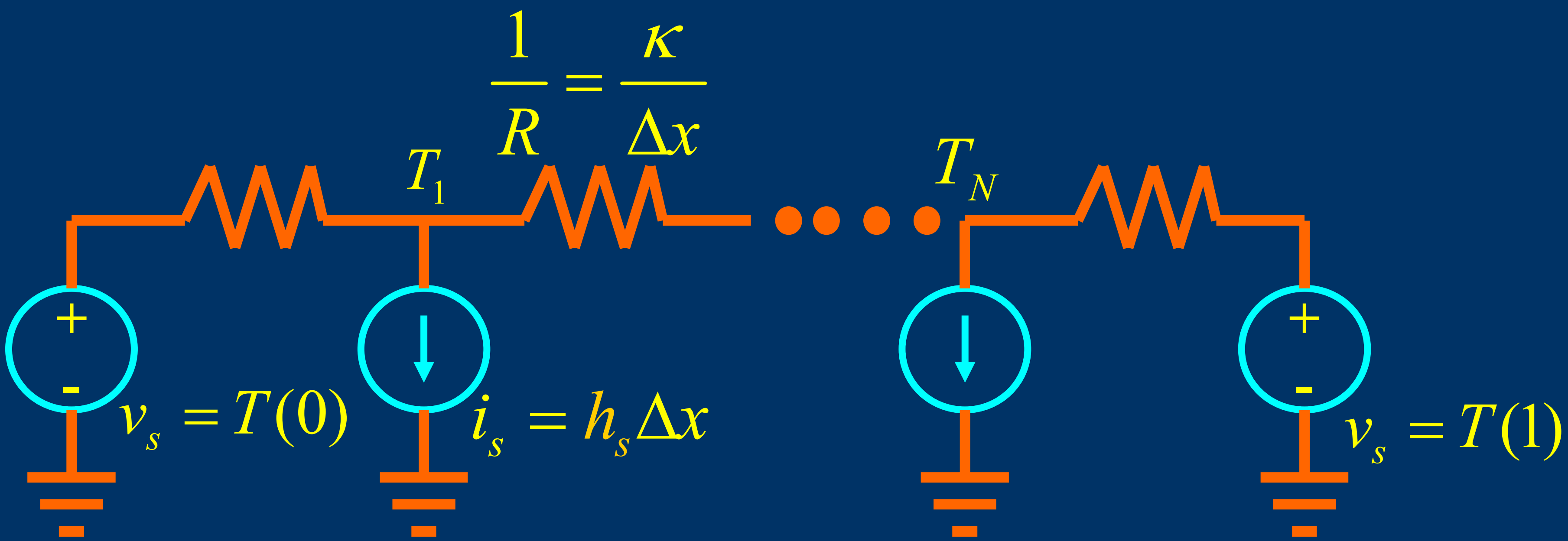
$$h_{i+1,i} - h_{i,i-1} = -h_s \Delta x$$

- $h_{i+1,i}$: Heat in from left
- $h_{i,i-1}$: Heat out from right
- h_s: **Incoming heat per unit length**

Limit as the sections become vanishingly small

$$\lim_{\Delta x \to 0} h_s(x) = \frac{\partial h(x)}{\partial x} = \frac{\partial}{\partial x} \kappa \frac{\partial T(x)}{\partial x}$$

Temperature analogous to Voltage
Heat Flow analogous to Current



Normalized Poisson Equation

$$\frac{\partial}{\partial x}\kappa\frac{\partial T(x)}{\partial x}=-h_s \Rightarrow -\frac{\partial^2 u(x)}{\partial x^2}=f(x)$$

$$-u_{xx}(x)=f(x)$$

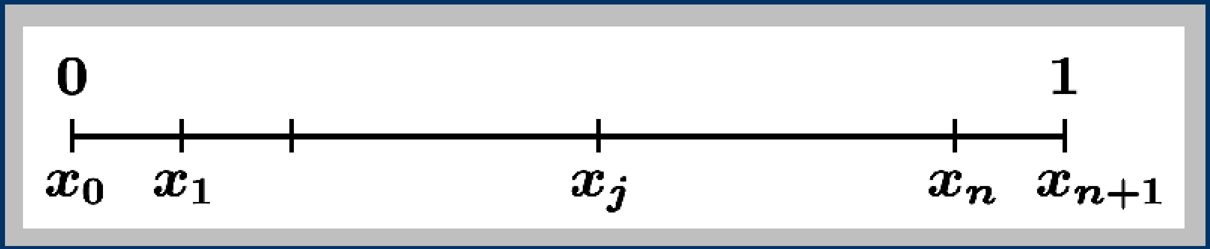
Numerical Solution

Finite Differences

Discretization

Subdivide interval $(0,1)$ into $n+1$ equal subintervals

$$\Delta x = \frac{1}{n+1}$$



$$x_j = j\Delta x, \qquad \hat{u}_j \approx u_j \equiv u(x_j)$$

for $\quad 0 \leq j \leq n+1$

Numerical Solution

Finite Differences

Approximation

For example . . .

$$v''(x_j) \approx \frac{1}{\Delta x}(v'(x_{j+1/2}) - v'(x_{j-1/2}))$$

$$\approx \frac{1}{\Delta x}(\frac{v_{j+1} - v_j}{\Delta x} - \frac{v_j - v_{j-1}}{\Delta x})$$

$$= \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2}$$

for Δx small

Using Basis Functions

Residual Equation

Partial Differential Equation form

$$-\frac{\partial^2 u}{\partial x^2} = f \qquad u(0)=0 \quad u(1)=0$$

Basis Function Representation

$$u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$$

Plug Basis Function Representation into the Equation

$$R(x) = \sum_{i=1}^{n} \omega_i \frac{d^2 \varphi_i(x)}{dx^2} + f(x)$$

Using Basis Functions

Example Basis functions

Introduce basis representation $u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

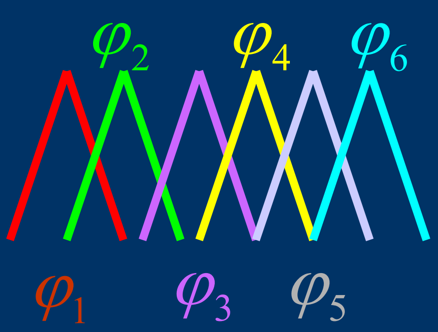
$\Rightarrow$ $u_h(x)$ is a weighted sum of basis functions

The basis functions define a space

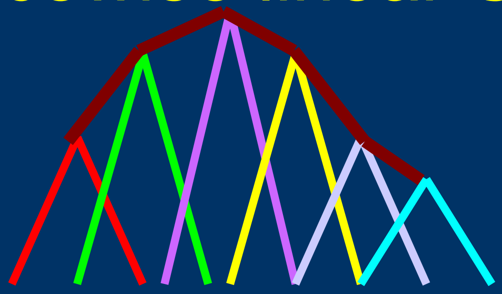
$$X_h = \left\{ v \in X_h \mid v = \sum_{i=1}^{n} \beta_i \varphi_i \ \text{ for some } \beta_i\text{'s} \right\}$$

Example

“Hat” basis functions



Piecewise linear Space



Using Basis functions

Basis Weights

Galerkin Scheme

Force the residual to be "orthogonal" to the basis functions

$$\int_0^1 \varphi_l(x) R(x) dt = 0$$

Generates n equations in n unknowns

$$\int_0^1 \varphi_l(x) \left[\sum_{i=1}^{n} \omega_i \frac{d^2 \varphi_i(x)}{dx^2} + f(x) \right] dx = 0 \quad l \in \{1, \ldots, n\}$$

Using Basis Functions

Basis Weights

Galerkin with integration by parts

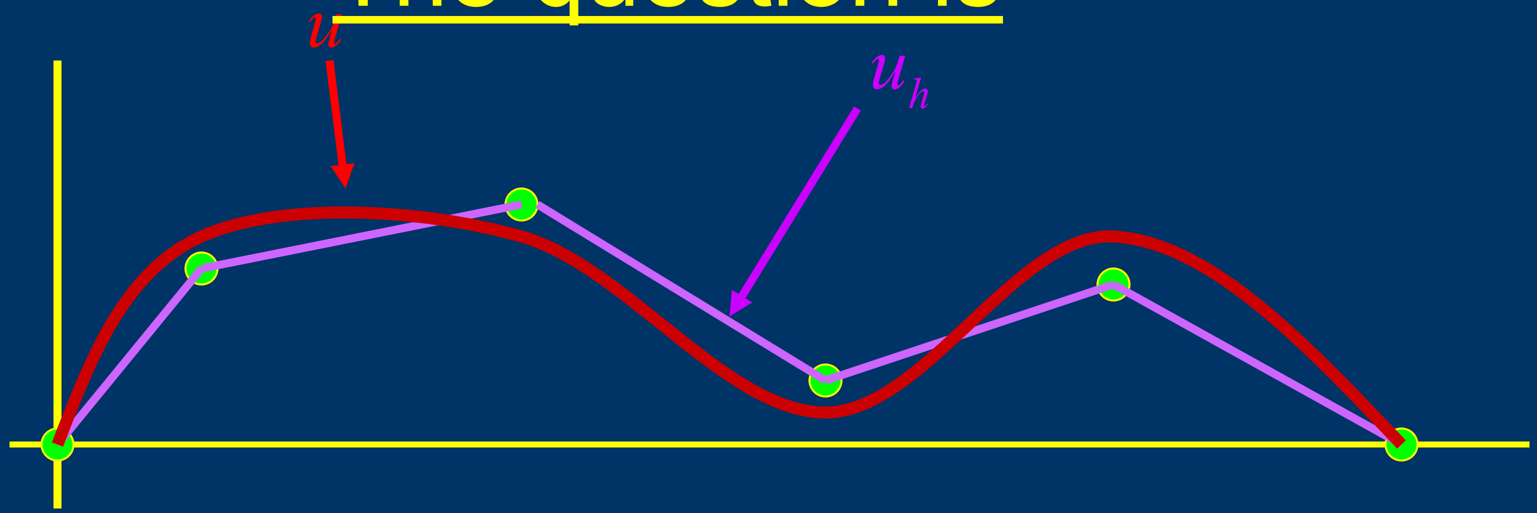
Only first derivatives of basis functions

$$\int_0^1 \frac{d\varphi_l(x)}{dx} \frac{d\sum_{i=1}^{n} \omega_i \varphi_i(x)}{dx} dx - \int_0^1 \varphi_i(x) f(x) dx = 0$$

$$l \in \{1, \ldots, n\}$$

Convergence Analysis

The question is



How does $\underbrace{\left\| u - u_h \right\|}_{error}$ decrease with refinement?

- This time – Finite-element methods
- Next time – Finite-difference methods

Heat Equation

Partial Differential Equation form

$$-\frac{\partial^2 u}{\partial x^2} = f \qquad u(0)=0 \quad u(1)=0$$

"Nearly" Equivalent weak form

$$\underbrace{\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx}_{a(u,v)} = \underbrace{\int_\Omega f\, v\, dx}_{l(v)} \quad \text{for all } v$$

Introduced an abstract notation for the equation u must satisfy

$$a(u,v) = l(v) \qquad \text{for all } v$$

Heat Equation

Convergence Analysis

Overview of FEM

Introduce basis representation $u(x) \approx u_h(x) = \sum_{i=1}^{n} \omega_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

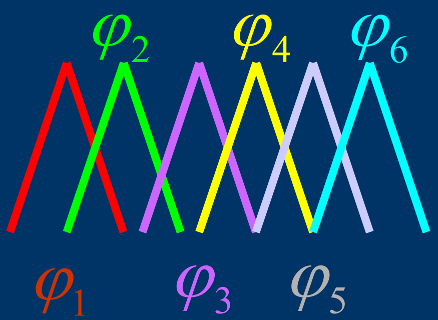
$\Rightarrow$ $u_h(x)$ is a weighted sum of basis functions

The basis functions define a space

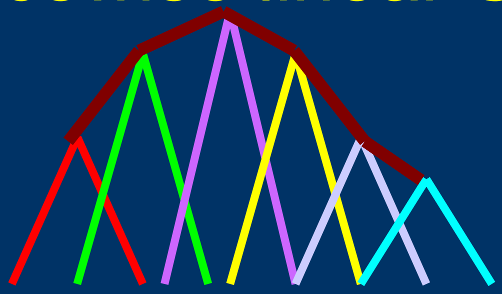
$$X_h = \left\{ v \in X_h \mid v = \sum_{i=1}^{n} \beta_i \varphi_i \ \text{ for some } \beta_i\text{'s} \right\}$$

Example

“Hat” basis functions



Piecewise linear Space



Key Idea

$a(u,u)$ defines a norm $\quad a(u,u) \equiv \left\|u\right\|$

U is restricted to be 0 at 0 and1!!

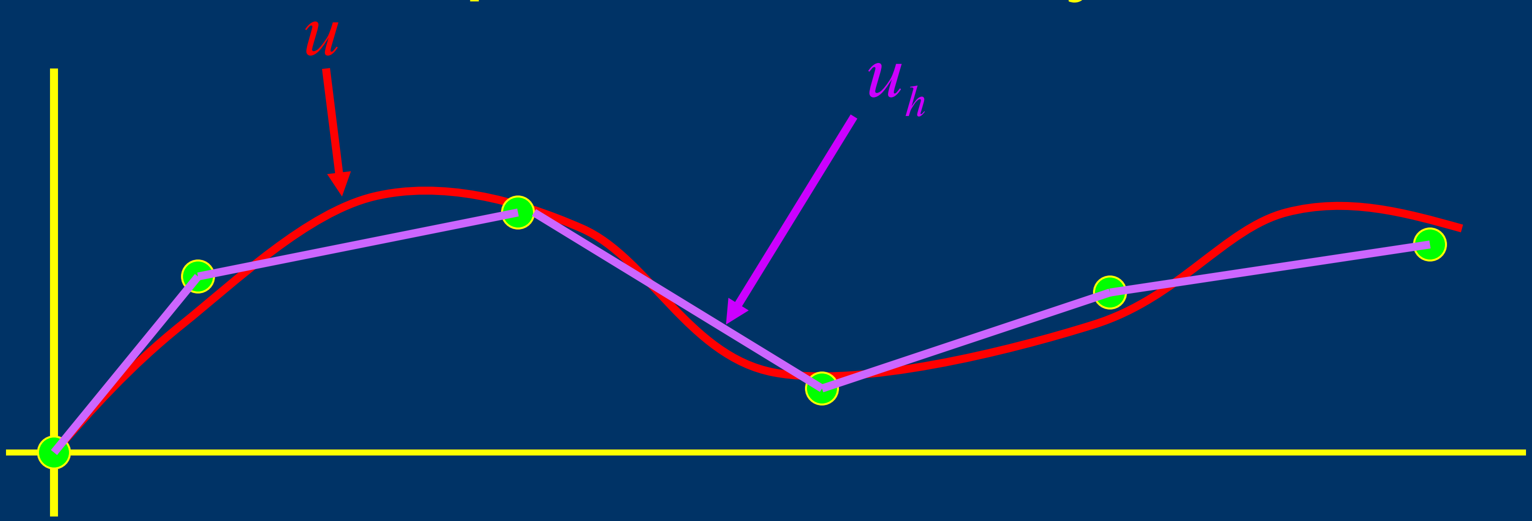
Using the norm properties, it is possible to show

If $\quad a(u_h,\varphi_i) = l(\varphi_i)$ for all $\varphi_i \in \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$

Then $\quad \underbrace{\left\|u - u_h\right\|}_{\text{Solution Error}} = \min_{w_h \in X_h} \underbrace{\left\|u - w_h\right\|}_{\text{Projection Error}}$

Heat Equation

The question is only



How well can you fit u with a member of X_h

But you must measure the error in the ||| ||| norm

For piecewise linear:
$$\underbrace{\left\| u - u_h \right\|}_{error} = O\left(\frac{1}{n}\right)$$

Problem of interest

Helmholtz Equation in 1D

Boundary Value Problem (BVP) - Strong Form

$$-u''(x) + \alpha\, u(x) = f(x) \qquad \alpha \geq 0$$

$$x \in (0,1),\ u(0) = u(1) = 0$$

Describes many physical phenomena (e.g.) :

- Temperature distribution in a bar $*$
- Deformation of an elastic bar N1
- Deformation of a string under tension N2

Problem of interest

Solution Properties

- the solution $u(x)$ always *exists*
- $u(x)$ is always smoother than the data $f(x)$
- given $f(x)$ the solution $u(x)$ is *unique*

Minimization Principle

Statement

Find

$$u = \arg\min_{w \in X} J(w)$$

where

$$X = \{v \text{ sufficiently smooth} \mid v(0) = v(1) = 0\},$$

and

$$J(w) = \frac{1}{2}\int_0^1 (w_x w_x + \alpha\, w w)\, dx - \int_0^1 f w\, dx$$

Minimization Principle

Statement

In words:

Over all functions w in X,

u that satisfies

$$-u_{xx} + \alpha\, u = f \quad \text{in } \Omega$$

$$u(0) = u(1) = 0$$

makes $J(w)$ as small as possible.

N4

Minimization Principle

Statement

Proof...

Let $w = u + v$.

Then

$$J(\overbrace{\underbrace{u}_{\in X} + \underbrace{v}_{\in X}}^{w \in X}) = \frac{1}{2}\int_0^1 (u+v)_x (u+v)_x \, dx$$

$$+ \frac{\alpha}{2}\int_0^1 (u+v)(u+v) \, dx$$

$$- \int_0^1 f(u+v) \, dx \, .$$

Minimization Principle

Statement

...Proof...

$$J(u+v) = \frac{1}{2}\int_0^1 (u_x u_x + \alpha\, uu)\, dx - \int_0^1 f u\, dx \qquad J(u)$$

$$+ \int_0^1 (u_x v_x + \alpha\, uv)\, dx - \int_0^1 f v\, dx \qquad \delta J_v(u)$$

first variation

$$+ \frac{1}{2}\int_0^1 (v_x v_x + \alpha\, vv)\, dx \qquad > 0 \text{ for } v \neq 0$$

Minimization Principle

Statement

...Proof...

$$\delta J_v(u) = \int_0^1 (u_x v_x + \alpha\, uv)\, dx - \int_0^1 fv\, dx$$

$$= \cancelto{0}{v}(0)\, u_x(0) - \cancelto{0}{v}(1)\, u_x(1) - \int_0^1 u_{xx} v\, dx$$

$$+ \alpha \int_0^1 uv\, dx - \int_0^1 fv\, dx$$

$$= \int_0^1 v \{ \underbrace{-u_{xx} + \alpha\, u - f}_{0} \}\, dx = 0, \ \forall v \in X$$

Minimization Principle

Statement

...Proof

$$J(\underbrace{u+v}_{w}) = J(u) + \underbrace{\frac{1}{2}\int_0^1 (v_x v_x + \alpha\, v v)\, dx}_{>\,0 \text{ unless } v\,=\,0}\,, \quad \forall\, v \in X$$

$\Rightarrow$

$$J(w) > J(u)\,, \qquad \forall\, w \in X\,,\ w \neq u$$

$$\Updownarrow$$

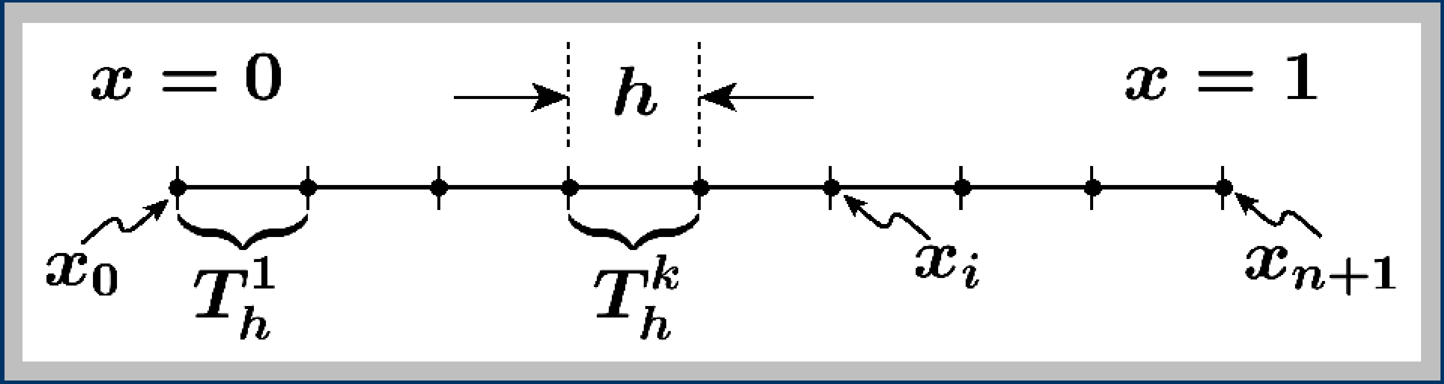
u is *the* minimizer of $J(w)$

E1 N5

Rayleigh-Ritz Approach

Approximation

Mesh



$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{T}_h^k$$

$T_h^k \, , \, k = 1, \ldots, K = n + 1$: *elements*

$x_i \, , \, i = 0, \ldots, n + 1$: *nodes*

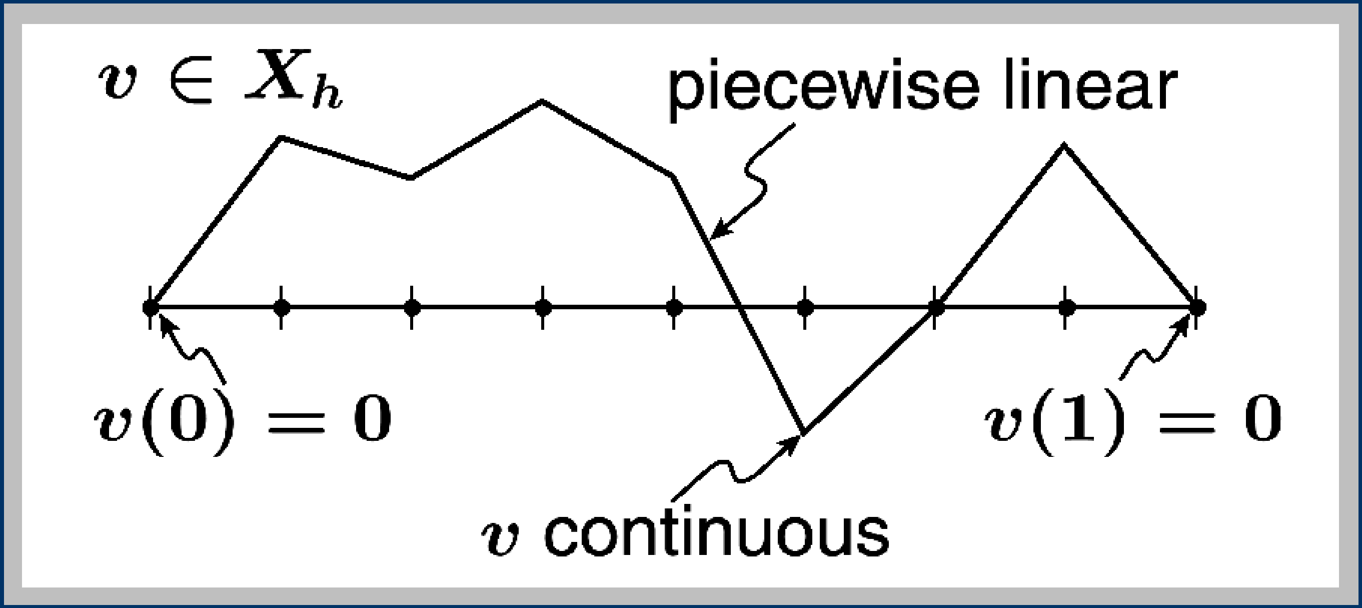
N6

Rayleigh-Ritz Approach

Approximation

Space $X_h \subset X$

$$X_h = \left\{ v \in X \;\middle|\; v|_{T_h^k} \in \mathbb{P}_1(T_h^k), \quad k = 1, \ldots, K \right\}$$



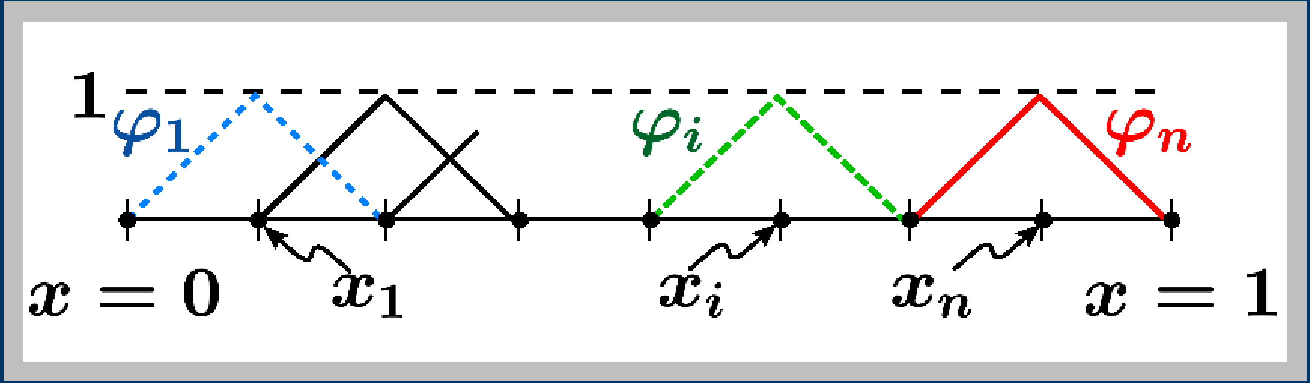
Rayleigh-Ritz Approach

Approximation

Basis

Nodal basis for X_h:

$$\varphi_j,\ j = 1, \ldots, n = \dim(X_h)$$



φ_i nonzero only on $\overline{T_h^i} \bigcup \overline{T_h^{i+1}}$

N7 N8

Rayleigh-Ritz Approach

"Projection"

Plan...

Let

$$\underbrace{u_h \quad (\in X_h)}_{\text{RR/FE Approximation}} = \sum_{j=1}^{n} u_{hj}\, \varphi_j(x) \; ;$$

set $u_{hj} = w_j$ that minimize

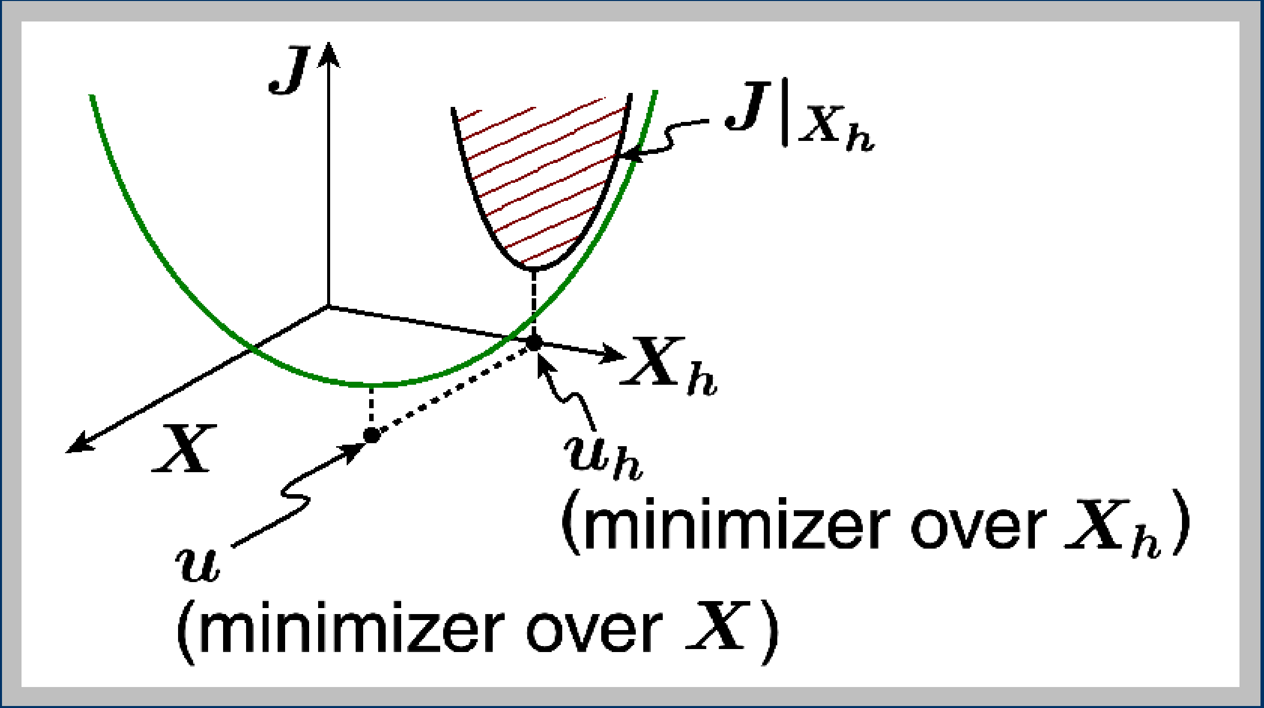
$$J\left(\sum_{j=1}^{n} w_j\, \varphi_j\right) \; .$$

Rayleigh-Ritz Approach

"Projection"

...Plan

Geometric Picture:



Rayleigh-Ritz Approach

"Projection"

$J|_{X_h} \cdots$

$$J\left(\sum_{j=1}^{n} w_j\,\varphi_j\right) = \frac{1}{2}\int_0^1 \frac{d}{dx}\Big(\sum_{i=1}^{n} w_i\,\varphi_i\Big)\frac{d}{dx}\Big(\sum_{j=1}^{n} w_j\,\varphi_j\Big)$$

$$+\frac{\alpha}{2}\int_0^1 \sum_{i=1}^{n}(w_i\,\varphi_i)\sum_{j=1}^{n}(w_j\,\varphi_j) - \int_0^1 f\sum_{j=1}^{n} w_j\,\varphi_j$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \int_0^1 \Big(\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + \alpha\varphi_i\varphi_j\Big)dx - \sum_{j=1}^{n} w_j \int_0^1 f\varphi_j\,dx$$

by *bilinearity* and *linearity*.

Rayleigh-Ritz Approach

"Projection"

$\ldots J|_{X_h}$

$$J^R(\underline{w} \in \mathbb{R}^n) \equiv J\left(\sum_{j=1}^{n} w_j \varphi_j\right)$$

$$= \frac{1}{2} \underline{w}^T \underline{A}_h \underline{w} - \underline{w}^T \underline{F}_h \,.$$

$$\underline{F}_h \in \mathbb{R}^n\!: F_{h\,i} = \int_0^1 f \, \varphi_i \, dx$$

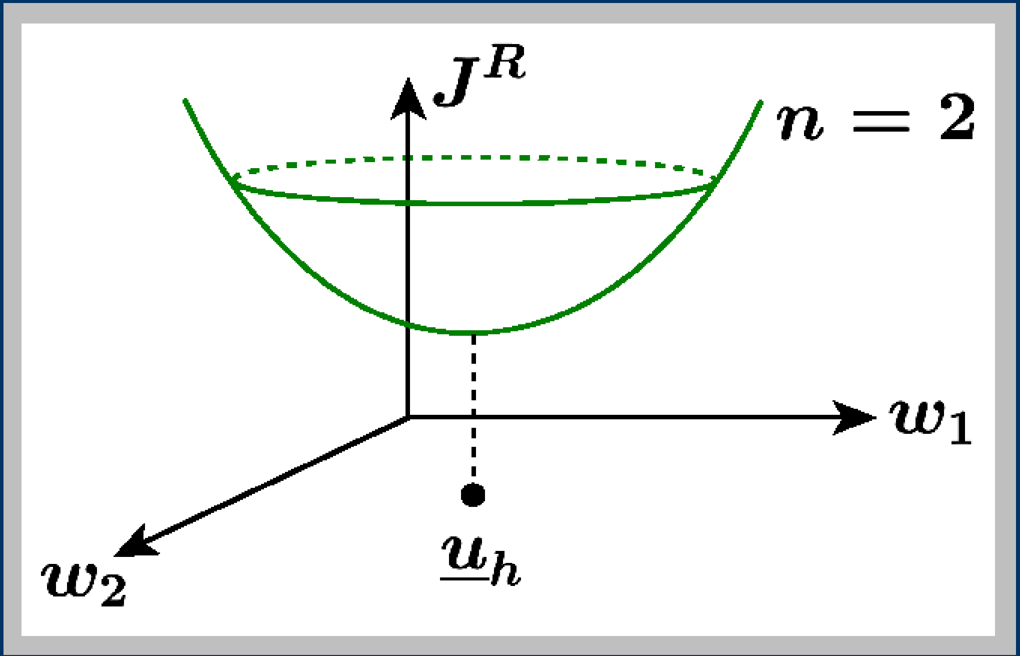
$$\underline{A}_h \in \mathbb{R}^{n \times n}\!: A_{h\,ij} = \int_0^1 \left(\frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + \alpha \, \varphi_i \varphi_j \right) dx$$

Rayleigh-Ritz Approach

"Projection"

Minimization...

$$\underline{u}_h = \arg \min_{\underline{w} \in \mathbb{R}^n} J^R(\underline{w})$$



Expand $J(\underline{w} = \underline{u}_h + \underline{v})$; require $J(\underline{w}) > J(\underline{u}_h)$ unless $\underline{v} = 0$.

Rayleigh-Ritz Approach

"Projection"

...Minimization...

$$J^R(\underline{u}_h + \underline{v})$$

$$= \frac{1}{2}(\underline{u}_h + \underline{v})^T \underline{A}_h(\underline{u}_h + \underline{v}) - (\underline{u}_h + \underline{v})^T \underline{F}_h$$

$$= \frac{1}{2}\underline{u}_h^T \underline{A}_h \underline{u}_h - \underline{u}_h^T \underline{F}_h$$

$$+ \frac{1}{2}\underline{v}^T \underline{A}_h \underline{u}_h + \frac{1}{2}\underline{u}_h^T \underline{A}_h \underline{v} - \underline{v}^T \underline{F}_h$$

$$+ \frac{1}{2}\underline{v}^T \underline{A}_h \underline{v}$$

Rayleigh-Ritz Approach

"Projection"

...Minimization...

$$J^R(\underline{u}_h + \underline{v}) = J(\underline{u})$$

$$+\underbrace{(\underline{A}_h\,\underline{u}_h - \underline{F}_h)^T}_{\nabla J^R(\underline{u}_h)}\,\underline{v} \quad \delta J^R_{\underline{v}}(\underline{u}_h)$$

*S*PD

$$+\frac{1}{2}\underbrace{\underline{v}^T\,\underline{A}_h\,\underline{v}}_{>0,\ \forall\,\underline{v}\neq 0}$$

S*PD*

Rayleigh-Ritz Approach

"Projection"

...Minimization

If (and only if)

$$\delta J_{\underline{v}}^{R}(\underline{u}_h) = 0, \quad \forall\, \underline{v} \in \mathbb{R}^n$$

$$\Updownarrow$$

$$\nabla J^{R}(\underline{u}_h) = \underline{A}_h\, \underline{u}_h - \underline{F}_h = \underline{0}$$

then

$$J(\underline{w} = \underline{u}_h + \underline{v}) > J(\underline{u}_h)\ , \qquad \forall\, \underline{v} \neq \mathbf{0}.$$

Rayleigh-Ritz Approach

"Projection"

Final Result

Find $\underline{u}_h \in \mathbb{R}^n$ such that

$$\underline{A}_h \, \underline{u}_h = \underline{F}_h \quad \Rightarrow \quad u_h(x) = \sum_{j=1}^{n} u_{hj} \, \varphi_j(x) \; .$$

SPD $\Rightarrow$ existence and uniqueness.

Error Analysis

Energy norm

Remember

$$J(u+v) = J(u) + \frac{1}{2}\underbrace{\int_0^1 (v_x\, v_x + \alpha\, v\, v)\, dx}_{\geq 0, SPD}, \ \forall v \in X$$

Define

$$|||v||| = \left[\int_0^1 (v_x\, v_x + \alpha\, v\, v)\, dx\right]^{\frac{1}{2}}$$

Energy norm

Error Analysis

Energy norm

Therefore

$$J(u+v) = J(u) + \frac{1}{2}|||v|||^2 \,, \ \forall v \in X$$

Choose any $w_h \in X_h \,, \ v \to (w_h - u) \in X$

$$J(w_h) = J(u) + \frac{1}{2}|||u - w_h|||^2 \,, \ \forall w_h \in X_h$$

For $w_h = u_h$

$$J(u_h) = J(u) + \frac{1}{2}|||u - u_h|||^2$$

Error Analysis

Energy norm

$$J(u_h) < J(w_h)\,,\ \forall w_h \in X_h\,,\ w_h \neq u_h$$

if $e = u - u_h$

$$|||\underbrace{u - u_h}_{e}||| < |||u - w_h|||\,,\ \forall w_h \in X_h\,,\ w_h \neq u_h$$

and

$$\boxed{|||e||| = \inf_{w_h \in X_h} |||u - w_h|||}$$

Error Analysis

Energy norm

In words: even if you *knew* u,

you could not find a w_h in X_h

more accurate than u_h

in the energy norm.

Error Analysis

A priori theory

A priori error estimates

N9

Energy norm:

$$|||e||| \leq C_1 \, h$$

L_2 norm:

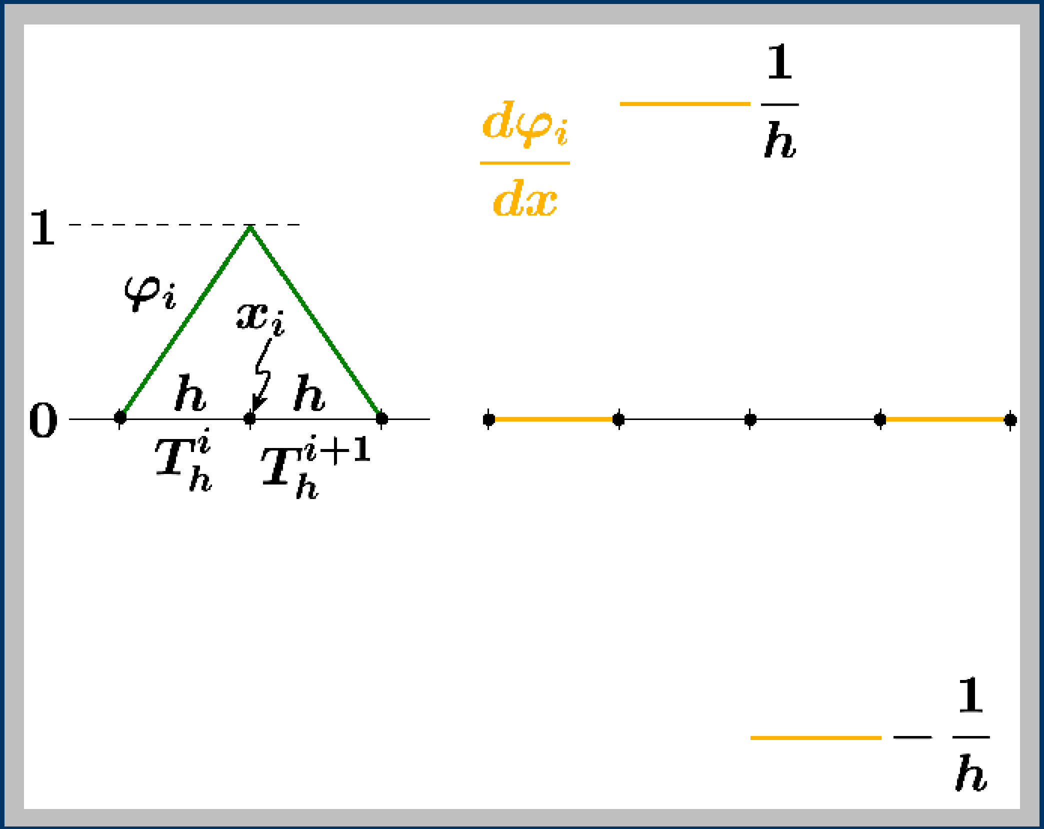
$$\|e\| = \left(\int_0^1 e \, e \, dx \right)^{1/2} \leq C_2 \, h^2$$

$C_{1,2} = \mathcal{F}(\Omega,$ problem parameters, smoothness of $u)$

Discrete Equations

Matrix Elements: $\underline{A}_h^1$

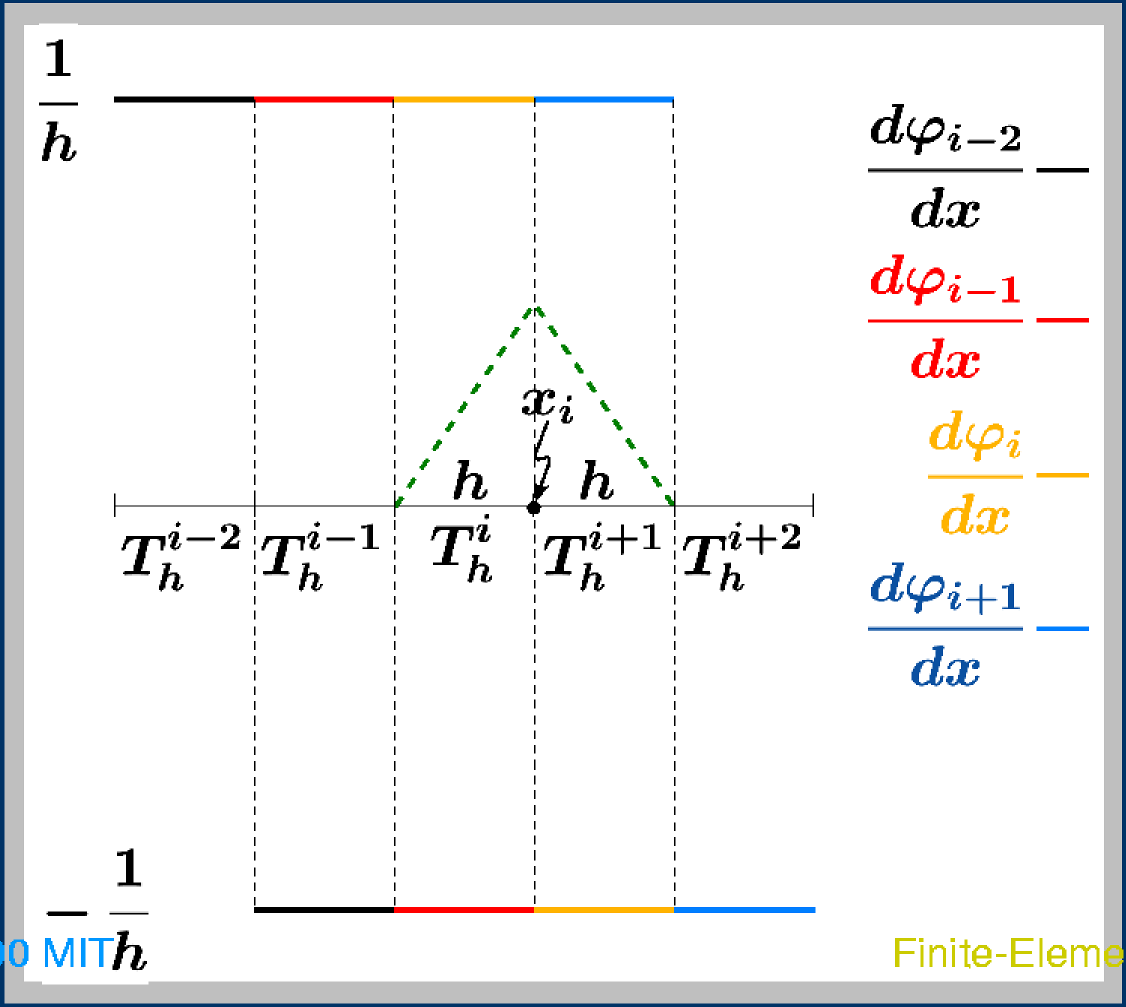
φ_i and $d\varphi_i/dx$...



Discrete Equations

Matrix Elements: $\underline{A}_h^1$

...φ_i and $d\varphi_i/dx$



Discrete Equations

Matrix Elements: $\underline{A}_h^1$

Typical Row

$$A^1_{h\,i\,j} = \int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx = \int_{T_h^i} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx + \int_{T_h^{i+1}} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}dx$$

is nonzero only for $i = j-1,\ j,\ j+1$

$$A^1_{h\,i\,i} = \frac{1}{h^2}\,(h) + \frac{1}{h^2}\,(h) = \frac{2}{h}$$

$$A^1_{h\,i\,i-1} = \frac{1}{h}\left(-\frac{1}{h}\right)(h) = -\frac{1}{h}$$

$$A^1_{h\,i\,i+1} = \left(-\frac{1}{h}\right)\frac{1}{h}\,(h) = -\frac{1}{h}$$

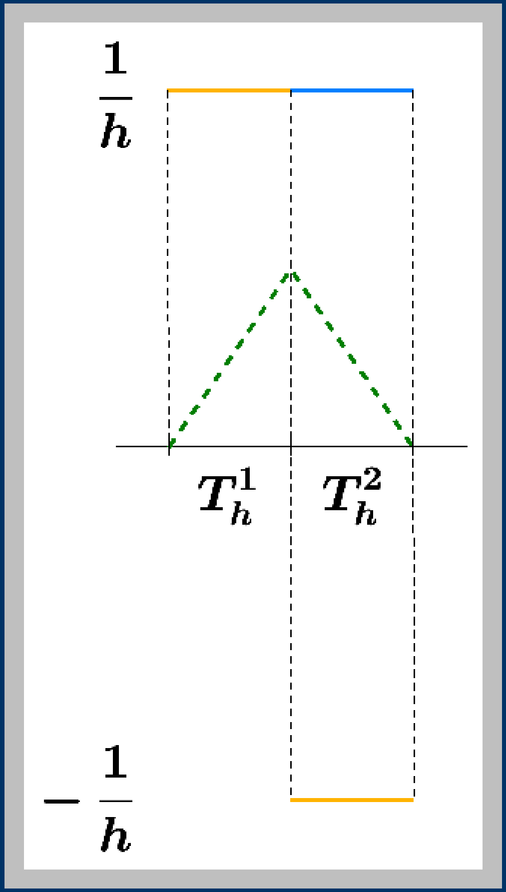
Discrete Equations

Matrix Elements: $\underline{A}_h^1$

Boundary Rows

$$A^1_{h\,1\,1} = \frac{2}{h},\ A^1_{h\,1\,2} = -\frac{1}{h},$$

$$A^1_{h\,n\,n} = \frac{2}{h},\ A^1_{h\,n\,n-1} = -\frac{1}{h}.$$



Discrete Equations

Matrix Elements: $\underline{A}_h^1$

Properties of $\underline{A}_h$

$$\underline{A}_h^1 = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & 0 & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$\underline{A}_h^1$ is SPD; *and*
diagonally dominant; *and*
sparse; *and*
tridiagonal.

Discrete Equations

Mass Matrix

$\underline{M}_h \in \mathbb{R}^{n \times n}$:

$$M_{h\,i\,j} = \int_0^1 \varphi_i\, \varphi_j\, dx$$

the finite element "identity" (I) operator

Is nonzero only for $i = j-1,\ j,\ j+1$

$$M_{h\,i\,j} = \int_{T_h^i} \varphi_i\, \varphi_j\, dx + \int_{T_h^{i+1}} \varphi_i\, \varphi_j\, dx$$

Discrete Equations

Mass Matrix

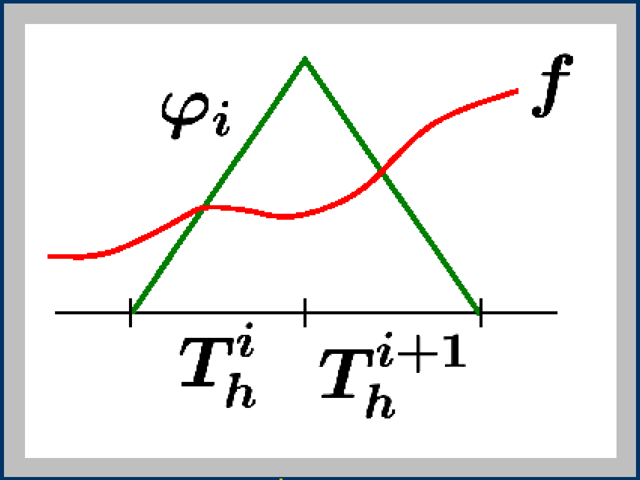
For linear elements, nodal basis:

$$\underline{M}_h = h \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & 0 \\ & & & \ddots & \\ & 0 & & \frac{2}{3} & \frac{1}{6} \\ & & & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

sparse, banded, tri-diagonal — “close” to $\underline{I}$.

Discrete Equations

"Load" Vector Elements: $\underline{F}_h$



$$F_{h\,i} = \int_0^1 f\,\varphi_i\,dx$$

$$F_{h\,i} = \int_{T_h^i} f\,\varphi_i\,dx + \int_{T_h^{i+1}} f\,\varphi_i\,dx, \quad i = 1, \ldots, n\,;$$

Discrete Equations

Summary

$\underline{u}_h \in \mathbb{R}^n$ satisfies

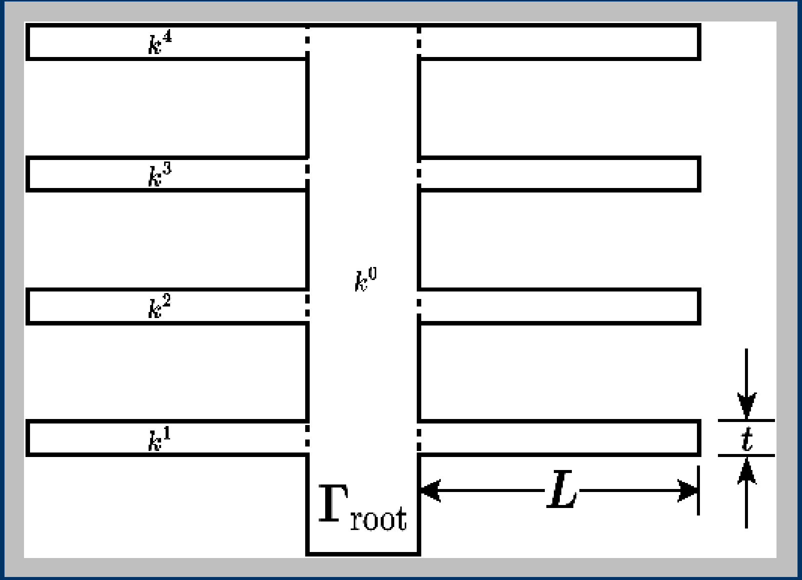
$$\left[\underline{A}_h^1 + \alpha \underline{M}_h\right] \begin{pmatrix} u_{h\,i} \\ \vdots \\ u_{h\,n} \end{pmatrix} = \begin{pmatrix} F_{h\,i} \\ \vdots \\ F_{h\,n} \end{pmatrix}$$

N10

Example

Heat Transfer Problem

Non-dimensional form



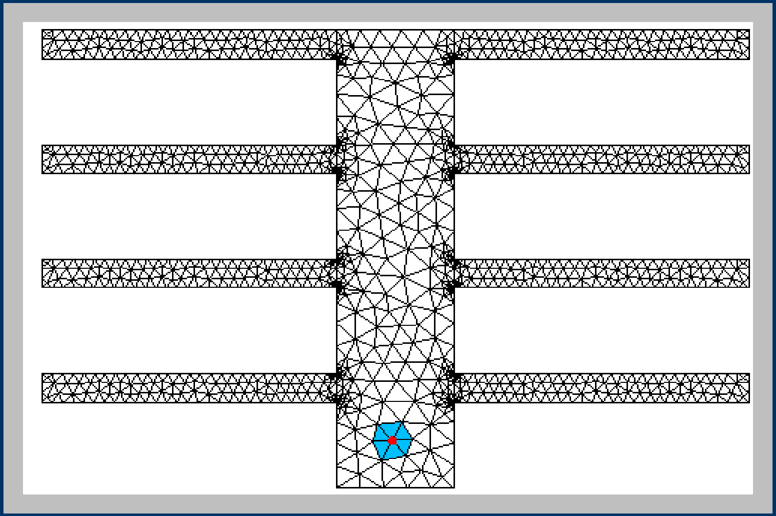
k^i: Thermal conductivity for $\Omega_i,\ i = 0, \ldots, 4$

Bi: Heat transfer coefficient

t, L : Geometric parameters

Finite element method

Example



$$X_h = span\{\varphi_1, \ldots, \varphi_n\}$$

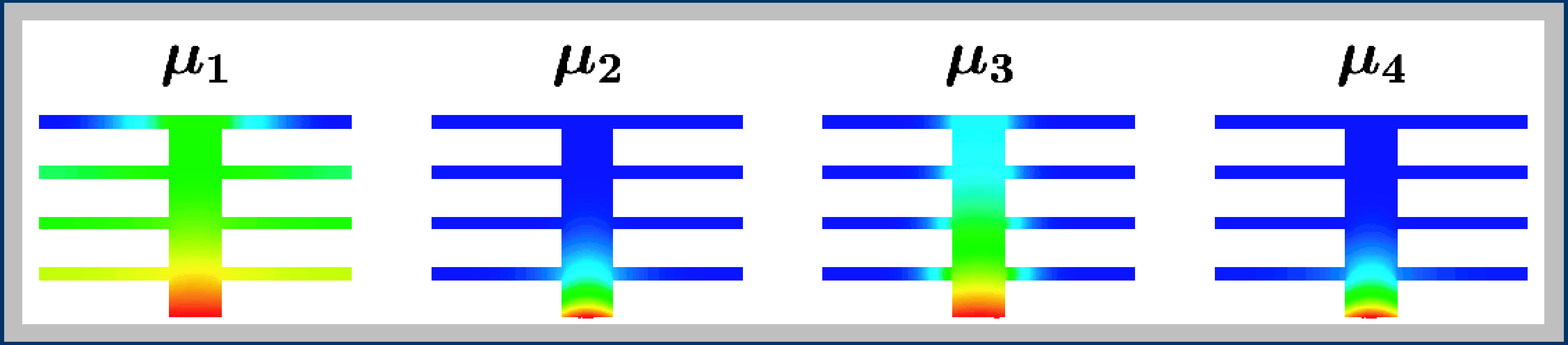
$\varphi_i(x)$:
Nodal basis functions

– First order elements

$$\boxed{\dim(X_h) = n}$$

Example

Possible solutions



Extensions

Example

- Complicated geometries
- General classes of problems
  (Good mathematical properties)
- Wider class of operators

Summary

- Why Poisson Equation
  - Reminder about heat conducting bar
- Finite-Difference And Basis function methods
  - Key question of convergence
- Convergence of Finite-Element methods
  - Key idea: solve Poisson by minimization
  - Demonstrate optimality in a carefully chosen norm