6.262: Discrete Stochastic Processes 4/6/11

Lecture 16: Renewals and Countable-state Markov

Outline:

- **Review major renewal theorems**
- **Age and duration at given t**
- **Countable-state Markov chains**

Sample-path time average (strong law for renewals)

$$\Pr\left\{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\overline{X}}\right\} = 1.$$

Ensemble & time average (elementary renewal thm)

$$\lim_{t\to\infty} \mathsf{E}\left[\frac{N(t)}{t}\right] = \frac{1}{\overline{X}}$$

Ensemble average (Blackwell's thm); $m(t) = \mathsf{E}[N(t)]$

$$\lim_{t\to\infty} [m(t+\lambda) - m(t)] = \frac{\lambda}{\overline{X}} \qquad \textbf{Arith. } X\textbf{, span } \lambda$$

$$\lim_{t\to\infty} [m(t+\delta) - m(t)] = \frac{\delta}{\overline{X}} \qquad \textbf{Non-Arith. } X\textbf{, any } \delta > 0$$

$$\lim_{t\to\infty}[m(t+\lambda)-m(t)] = \frac{\lambda}{\overline{X}} \qquad \textbf{Arith. } X\textbf{, span } \lambda$$

can be rewritten as

$$\lim_{n\to\infty}\Pr\{\textbf{renewal at } n\lambda\} = \frac{\lambda}{\overline{X}} \qquad \textbf{Arith. } X\textbf{, span } \lambda$$

If we model an arithmetic renewal process as a Markov chain starting in the renewal state 0, this essentially says $P_{00}^n \to \pi_0$**.**

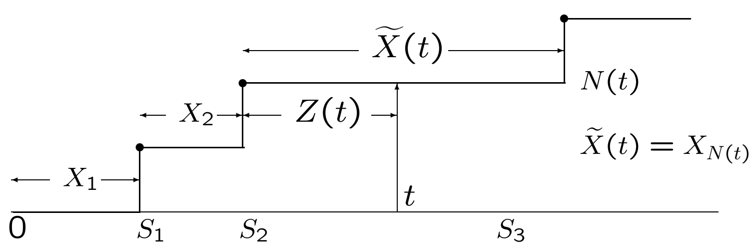
$$\lim_{t\to\infty}[m(t+\delta)-m(t)] = \frac{\delta}{\overline{X}} \qquad \textbf{Non-Arith. } X\textbf{, any } \delta > 0$$

This is the best one could hope for. Note that

$$\lim_{\delta\to 0}\lim_{t\to\infty}\frac{m(t+\delta)-m(t)}{\delta} = \frac{1}{\overline{X}}$$

but the order of the limits can't be interchanged.

Age and duration at given t



Assume an arithmetic renewal process of span 1.

For integer t**,** $Z(t) = i \ge 0$ **and** $\widetilde{X}(t) = k > i)$ **iff there are successive arrivals at** $t-i$ **and** $t-i+k$**.**

Let $q_j = \Pr\{\textbf{arrival at time } j\} = \sum_{n\ge 1}\mathsf{p}_{S_n}(j)$ **and let** $q_0 = 1$ **(nominal arrival at time 0). Then**

$$\mathsf{p}_{Z(t),\widetilde{X}(t)}(i,k) = q_{t-i}\mathsf{p}_X(k) \qquad \textbf{for } 0 \le i \le t;\ k > i$$

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Note that

$q_i = \Pr\{$**arrival at** $j\} = \mathsf{E}\,[$**arrival at** $j] = m(i) - m(i-1),$

so by Blackwell, $\lim_{j\to\infty} 1/\overline{X}$.

$$\lim_{t\to\infty} \mathsf{p}_{Z(t),\widetilde{X}(t)}(i,k) \;=\; \frac{\mathsf{p}_X(k)}{\overline{X}} \qquad \textbf{for } k > i \ge 0.$$

$$\lim_{t\to\infty} \mathsf{p}_{Z(t)}(i) \;=\; \frac{\sum_{k=i+1}^{\infty} \mathsf{p}_X(k)}{\overline{X}} \;=\; \frac{\mathsf{F}^c_X(i)}{\overline{X}} \quad \textbf{for } i \ge 0.$$

$$\lim_{t\to\infty} \mathsf{p}_{\widetilde{X}(t)}(k) \;=\; \frac{\sum_{i=0}^{k-1} \mathsf{p}_X(k)}{\overline{X}} \;=\; \frac{k\mathsf{p}_X(k)}{\overline{X}} \quad \textbf{for } k \ge 1.$$

Now look at asymptotic expected duration:

$$\lim_{t\to\infty} \mathsf{E}\left[\widetilde{X}(t)\right] = \sum_{k=1}^{\infty} k \cdot k\mathsf{p}_X(k)/\overline{X} \;=\; \mathsf{E}\left[X^2\right]/\overline{X}$$

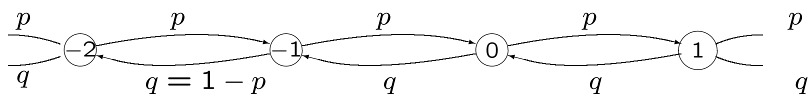
This is the same as the sample-path average, but now we can look at the finite t case. More important, we get a different interpretation.

For a given $\widetilde{X} = k$, there are k equiprobable choices for age; for each choice, the joint $Z, \widetilde{X}$ PMF is $\mathsf{p}_X(k)/\overline{X}$. Thus large durations are enhanced relative to inter-renewals.

The expected age (after some work) is $\mathsf{E}\left[X^2\right]/2\overline{X} - \frac{1}{2}$. This is at integer values of large t. The age increases linearly with slope 1 to the next integer value and then drops by 1.

Countable -state Markov chains

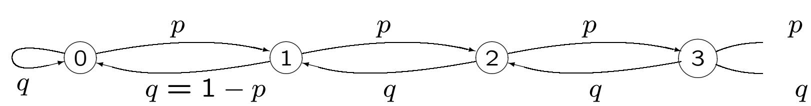
The biggest change from finite-state Markov chains to countable-state chains is the concept of a recurrent class. Example:



This Markov chain models a Bernoulli ± 1 process. The state at time n is $S_n = X_1 + X_2 + \cdots X_n$. The state S_n at time n is $j = 2k - n$ where k is the number of positive transitions in the n trials.

All states communicate and have a period $d = 2$; $\sigma^2_{S_n} = n[1-(p-q)^2]$. $P^n_{0,j}$ approaches 0 at least as $1/\sqrt{n}$ for every j.

Another example (called a birth-death chain)



In this case, if $p > 1/2$, the state drifts to the right and P^n_{0j} approaches 0 for all j. If $p < 1/2$, it drifts to the left and keeps bumping state 0.

A truncated version of this was analyzed in the homework. With $p > 1/2$, the steady-state increases to the right, with $p < 1/2$, it increases to the left, and at $p = 1/2$ it is uniform.

As the truncation point increases, the 'steady-state' remains positive only for $p < 1/2$.

We want to define recurrent to mean that, given $X_0 = i$, there is a future return to state i WP1. We will see that the birth-death chain above is recurrent if $p < 1/2$ and not recurrent if $p > 1/2$. The case $p = 1/2$ is strange and will be called null-recurrent.

We can use renewal theory to study recurrent chains, but first must understand first-passage-times.

Def: The first-passage-time probability, $f_{ij}(n)$, is

$$f_{ij}(n) = \Pr\{X_n{=}j, X_{n-1}{\neq}j, X_{n-2}{\neq}j, \ldots, X_1{\neq}j | X_0{=}i\}.$$

It's the probability, given $X_0 = i$, that n is the first epoch at which $X_n = j$. Then

$$f_{ij}(n) = \sum_{k \neq j} P_{ik} f_{kj}(n-1); \qquad n > 1; \qquad\qquad f_{ij}(1) = P_{ij}.$$

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Recall that Chapman-Kolmogorov says

$$P_{ij}^n = \sum_k P_{ik} P_{kj}^{n-1},$$

so the difference between $f_{ij}(n)$ and P_{ij}^n is only in cutting off the outputs from j (as before in finding expected first-passage-times.

Let $\mathsf{F}_{ij}(n) = \sum_{m \leq n} f_{ij}(m)$ be the probability of reaching j by time n or before. If $\lim_{n\to\infty} \mathsf{F}_{ij}(n) = 1$, there is a rv T_{ij} with distribution function F_{ij} that is the first-passage-time rv.

We can also express $\mathsf{F}_{ij}(n)$ as

$$\mathsf{F}_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik}\mathsf{F}_{kj}(n-1); \quad n > 1; \quad \mathsf{F}_{ij}(1) = P_{ij}$$

Since $F_{ij}(n)$ is nondecreasing in n, the limit $F_{ij}(\infty)$ must exist and satisfy

$$\mathsf{F}_{ij}(\infty) = P_{ij} + \sum_{k \neq j} P_{ik}\mathsf{F}_{kj}(\infty).$$

Unfortunately, choosing $\mathsf{F}_{ij}(\infty) = 1$ for all i, j satisfies these equations. The correct solution turns out to be the smallest set of $\mathsf{F}_{ij}(\infty)$ that satisfies these equations.

If $\mathsf{F}_{jj}(\infty) = 1$, then an eventual return from state j occurs with probability 1 and the sequence of returns is the sequence of renewal epochs in a renewal process.

If $\mathsf{F}_{jj}(\infty) = 1$, then there is a rv T_{jj} with the distribution function $\mathsf{F}_{jj}(n)$ and j is recurrent. The renewal process of returns to j then has inter-renewal intervals with the distribution function $\mathsf{F}_{jj}(n)$.

From renewal theory, the following are equivalent:

1) state j is recurrent.

2) $\lim_{t\to\infty} N_{jj}(t) = \infty$ with probability 1.

3) $\lim_{t\to\infty} \mathsf{E}\left[N_{jj}(t)\right] = \infty$.

4) $\lim_{t\to\infty} \sum_{1\leq n\leq t} P_{jj}^{n} = \infty$.

None of these imply that $\mathsf{E}\left[T_{jj}\right] < \infty$.

Two states are in the same class if they communicate (same as for finite-state chains).

If states i and j are in the same class then either both are recurrent or both transient (not recurrent).

Pf: If j is recurrent, then $\sum_n P_{jj}^n = \infty$. Then

$$\sum_{n=1}^{\infty} P_{ii}^n \geq \sum_{k=1}^{\infty} P_{ij}^m P_{jj}^k P_{jk}^\ell = \infty$$

All states in a class are recurrent or all are transient.

By the same kind of argument, if i, j are recurrent, then $\mathsf{F}_{ij}(\infty) = 1$.

If a state j is recurrent, then T_{jj} might or might not have a finite expectation.

Def: If $\mathsf{E}\left[T_{jj}\right] < \infty$, j is positive recurrent. If T_{jj} is a rv and $\mathsf{E}\left[T_{jj}\right] = \infty$, then j is null recurrent. Otherwise j is transient.

For $p = 1/2$, each state in each of the following is null recurrent.