Solutions to Homework 9

6.262 Discrete Stochastic Processes
MIT, Spring 2011

Exercise 5.6:

Let $\{X_n; n \geq 0\}$ be a branching process with $X_0 = 1$. Let $\overline{Y}$, σ^2 be the mean and variance of the number of offspring of an individual.

a) Argue that $\lim_{n\to\infty} X_n$ exists with probability 1 and either has the value 0 (with probability $\mathsf{F}_{10}(\infty)$) or the value ∞ (with probability $1 - \mathsf{F}_{10}(\infty)$).

Solution 5.6a We consider 2 special, rather trivial, cases before considering the important case (the case covered in the text). Let p_i be the PMF of the number of offspring of each individual. Then if $p_1 = 1$, we see that $X_n = 1$ for all n, so the statement to be argued is simply false. It is curious that this exercise has been given many times over the years with no one pointing this out.

The next special case is where $p_0 = 0$ and $p_1 < 1$. Then $X_{n+1} \geq X_n$ (*i.e.*, the population never shrinks but can grow). Since $X_n(\omega)$ is nondecreasing for each sample path, either $\lim_{n\to\infty} X_n(\omega) = \infty$ or $\lim_{n\to\infty} X_n(\omega) = j$ for some $j < \infty$. The latter case is impossible, since $P_{jj} = p_1^j$ and thus $P_{jj}^m = p_1^{mj} \to 0$.

Ruling out these two trivial cases, we have $p_0 > 0$ and $p_1 < 1 - p_0$. In this case, state 0 is recurrent (*i.e.*, it is a trapping state) and states $1, 2, \dots,$ are in a transient class. To see this, note that $P_{10} = p_0 > 0$, so $\mathsf{F}_{11}(\infty) \leq 1 - p_0 < 1$, which means by definition that state 1 is transient. All states $i > 1$ communicate with state 1, so by Theorem 5.1.1, all states $j \geq 1$ are transient. Thus one can argue that the process has 'no place to go' other than 0 or ∞.

The following ugly analysis makes this precise. Note from Lemma 5.1.1 part 4 that

$$\lim_{t\to\infty} \sum_{n\leq t} P_{jj}^t \neq \infty.$$

Since this sum is nondecreasing in t, the limit must exist and the limit must be finite This means that

$$\lim_{t\to\infty} \sum\nolimits_{n\geq t} P_{jj}^n = 0$$

Now we can write $P_{1j}^n = \sum_{\ell\leq n} f_{1j}^\ell P_{jj}^{n-\ell}$, from which it can be seen that $\lim_{t\to\infty} \sum_{n\geq t} P_{1j}^n = 0$.

From this, we see that for every finite integer ℓ,

$$\lim_{t\to\infty} \sum_{n\geq t} \sum_{j=1}^{\ell} P_{1j}^n = 0$$

This says that for every $\epsilon > 0$, there is a t sufficiently large that the probability of ever entering states 1 to ℓ on or after step t is less than ϵ. Since $\epsilon > 0$ is arbitrary, all sample paths (other than a set of probability 0) never enter states 1 to ℓ after some finite time.

Since ℓ is arbitrary, $\lim_{n\to\infty} X_n$ exists WP1 and is either 0 or ∞. By definition, it is 0 with probability $F_{10}(\infty)$.

b) Since X_n is the sum of a random number (X_{n-1}) of IID random variables each of mean $\overline{Y}$ and variance σ^2, we have

$$\begin{aligned} Var(X_n) &= \mathbb{E}[X_{n-1}]\sigma^2 + \overline{Y}^2 Var(X_{n-1}) \\ &= \mathbb{E}[X_0]\overline{Y}^{n-1}\sigma^2 + \overline{Y}^2 Var(X_{n-1}) \\ &= \overline{Y}^{n-1}\sigma^2 + \overline{Y}^2 Var(X_{n-1}) \end{aligned}$$

Where we used the facts that $\mathbb{E}[X_{n-1}] = \mathbb{E}[X_0]\overline{Y}^{n-1}$ and $X_0 = 1$. For $\overline{Y} \neq 1$, we use induction on n to establish the desired result.The basic step $(n = 1)$ is

$$Var(X_1) = \sigma^2 = \sigma^2\overline{Y}^{n-1}\frac{\overline{Y}^n - 1}{\overline{Y} - 1}, \text{ for } n = 1.$$

Assume that $Var(X_{n-1}) = \sigma^2\overline{Y}^{n-2}(\overline{Y}^{n-1} - 1)/(\overline{Y} - 1)$. Then from the recurrence equation

$$\begin{aligned} Var(X_n) &= \overline{Y}^{n-1}\sigma^2 + \overline{Y}^2\{\sigma^2\overline{Y}^{n-2}(\overline{n-1} - 1)/(\overline{Y} - 1)\} \\ &= \sigma^2\overline{Y}^{n-1}(\overline{Y}^n - 1)/(\overline{Y} - 1). \end{aligned}$$

Which completes the inductive argument. For $\overline{Y} = 1$, we have $Var(X_n) = \sigma^2 + Var(X_{n-1}) = 2\sigma^2 + Var(X_{n-2}) = n\sigma^2 + Var(X_0) = n\sigma^2$.

Exercise 5.7:

Using theorem 5.3.2, we will show that the chain is reversible by demonstrating a set $\{\pi_i\}$ of steady state probabilities for which $\pi_i P_{ij} = \pi_j P_{ji}$ for all i, j. Thus, we want to find $\{\pi_i\}$ satisfying

$$\pi_i \frac{d_{ij}}{\sum_k d_{ik}} = \pi_j \frac{d_{ij}}{\sum_k d_{jk}} \tag{1}$$

Where we have used $d_{ij} = d_{ji}$ in the upper right corner of the above equation. This equation will be satisfied if we choose π_i to be proportional to $\sum_k d_{ik}$. Normalizing $\{\pi_i\}$ to satisfy $\sum_i \pi_i = 1$, we see that eq. (1) is satisfied by

$$\pi_i = \frac{\sum_k d_{ik}}{\sum_j \sum_k d_{jk}}$$

Exercise 5.8:

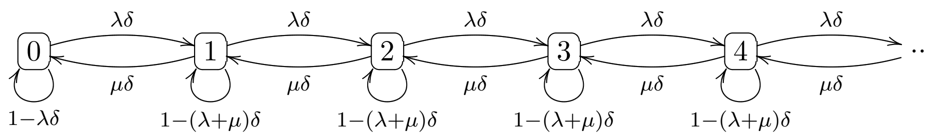
Note that if $\overline{\pi}_i$ is summed over i, the numerator term becomes the same as the denominator, so that $\sum_i \overline{\pi}_i = 1$. Thus, using theorem 5.3.2., it suffices to show that $\overline{\pi}_i P'_{ij} = \overline{\pi}_j P'_{ji}$. We have

$$\overline{\pi}_i P'_{ij} = \frac{\pi_i P_{ij}}{\sum_{k=0}^{M} \pi_k \sum_{m=0}^{M} P_{km}}$$

Since the denominator is independent of i and j, and since the reversibility of the original chain implies that $\pi_i P_{ij} = \pi_j P_{ji}$, we have the desired result.

Exercise 5.10:

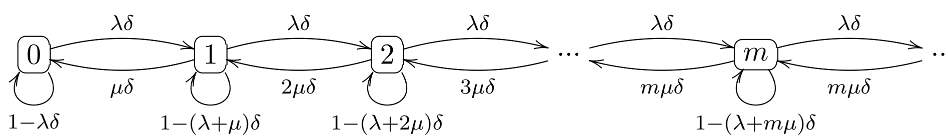
a) M/M/1:



From (5.40), we have

$$\pi_i = \rho^i(1-\rho), \text{ for } i \geq 0 \text{ where } \rho = \lambda/\mu \text{ and } \rho < 1 \text{ (positive recurrent).}$$

M/M/m:



Using the steady-state equations (5.25) and defining $\rho = \lambda/(m\mu)$, we have

$$\begin{aligned} \pi_i/\pi_{i-1} &= \lambda/(i\mu), &&\text{for } i < m; \\ \pi_i/\pi_{i-1} &= \rho, &&\text{for } i \geq m; \end{aligned}$$

$$\begin{aligned} \pi_i/\pi_{i-1} &= (\lambda/\mu)^i \pi_0/i, &&\text{for } i < m; \\ \pi_i/\pi_{i-1} &= \rho^i \pi_0 m^m/m!, &&\text{for } i \geq m; \end{aligned}$$

Since $\sum_i \pi_i = 1$, we have

$$\pi_0 = \left[1 + \sum_{i=1}^{m-1} (\lambda/\mu)^i/i! + \sum_{i=m}^{\infty} \rho^i m^m/m!\right]^{-1} = \left[1 + \sum_{i=1}^{i=m-1} (\lambda/\mu)^i/i! + \frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$$

M/M/∞: Setting $m = \infty$ in the M/M/m result, we get:

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{\pi_0}{i!}, \text{ for all } i \geq 0$$

Using the Taylor sries expansion of $e^{\lambda/mu} = \sum_i (\lambda/\mu)^i/i!$, we see that $\pi_0 = e^{-\lambda/\mu}$. Thus,

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{\exp(-\lambda/\mu)}{i!}, \text{ for all } i \geq 0$$

b) M/M/1: For the chain to be transient, we need $\lambda/\mu > 1$, for null recurrent, $\lambda/\mu = 1$, and for positive recurrent $\lambda/\mu < 1$.

M/M/m: For the chain to be transient, we need $\lambda/m\mu > 1$, for null-recurrent, $\lambda/m\mu = 1$, for positive recurrent $\lambda/m\mu < 1$.

M/M/∞: For the chain to be transient, we need $\lambda > 0$ and $\mu = 0$ (i.e., customers arrive but they do not depart.) We can not have null-recurrence. For $\mu > 0$, we show that the expected queue length is finite, which implies steady state probabilities. For the chain to be positive recurrent, $\mu > 0$.

c) Assume positive recurrence for each queue.
M/M/1: $L = \sum_i i\pi_i = \rho/(1-\rho) = \lambda/(\mu - \lambda)$. To find L_q, we observe that L is L_q plus the expected number of customers in the service, i.e., $L = L_q + (a - \pi_0)$. Thus,

$$L_q = L - (1 - \pi_0) = \rho^2/(1-\rho) = \lambda^2/[\mu(\mu - \lambda)].$$

Using Little's theorem,

$$W = L/\lambda = a/[\mu(1-\rho)] = 1/(\mu - \lambda)$$

$$W_q = L_q/\lambda = \rho/[\mu(1-\rho)] = \lambda/[\mu(\mu - \lambda)]$$

M/M/m:
There are customers in the queue only if all the servers are busy, i.e., if there are more customers than servers in the system:

$$L_q = \sum_{i>m} (i-m)\pi_i = \sum_{i>m} (i-m)\rho^i \pi_0 m^m/m! = \rho\pi_0(\rho m)^m/[(1-\rho)^2 m!]$$

Where π_0 is given in part (a). The expected delay W_q in the queue is then given by Little's formula of a customer in the system as $W_q = L_q/\lambda$. The delay W in the system is the queuing delay plus service delay, so $W = L_q/\lambda + 1/\mu$. Finally, the expected number in the system is given by Little's law again as $L = W\lambda = L_q + \lambda/\mu$. Thus, in terms of L_q,

$$\begin{aligned} L &= L_q + \lambda/\mu \\ W &= L_q/\lambda + 1/\mu \\ W_q &= L_q/\lambda \end{aligned}$$

<u>M/M/∞</u>: There are no customers waiting for service, so $L_q = W_q = 0$. Each customer waits in the system for its own service time, so $W = 1/\mu$. By Little's formula, $L = \lambda/\mu$.

<u>**Exercise 6.1:**</u>

a) The holding interval U_1 conditional on $X_0 = i$ is exponentially distributed with parameter v_i. And v_i is uniquely determined by transition rates q_{ij} as:

$$v_i = \sum_j q_{ij} = q_{i,i+1} + q_{i,i-1} = \lambda + \mu$$

Thus, $\mathbb{E}[U_1|X_0 = i] = 1/v_i = 1/(\lambda+\mu)$.

b) The holding interval U_n between the time that state $X_{n-1} = l$ is entered and X_n entered, conditional on X_{n-1} is jointly independent of X_m for all $m \neq n-1$. So, $\mathbb{E}[U_1|X_0 = i, X_1 = i+1] = \mathbb{E}[U_1|X_0 = i] = 1/v_i = 1/(\lambda+\mu)$.

The same is true for $\mathbb{E}[U_1|X_0 = i, X_1 = i+1]$.

c) Conditional on $\{X_0 = i, X_{i+1} = i+1\}$, we know that the first transition is an arrival, so the first arrival time (V) is the same as the first holding interval (U). Thus,

$$\mathbb{E}[V|X_0 = i, X_1 = i+1] = \mathbb{E}[U_1|X_0 = i, X_1 = i+1] = 1/(\lambda+\mu)$$

Conditional on $\{X_0 = i, X_{i+1} = i-1\}$, we know that the first transition is a departure.So the time until the first arrival is sum of the time for first transition (i.e., a departure) and the time until the next arrival. The second term is exponentially distributed with rate λ, so we have:

$$\mathbb{E}[V|X_0 = i, X_1 = i-1] = \mathbb{E}[U_1|X_0 = i, X_1 = i-1] + \mathbb{E}[V|X_1 = i-1] = \frac{1}{\lambda+\mu} + \frac{1}{\lambda}$$

d) Using the total expectation lemma, we have:

$$\begin{aligned}
\mathbb{E}[V|X_0 = i] &= \mathbb{E}[V|X_0 = i, X_1 = i+1]\Pr\{X_1 = i+1|X_0 = i\} + \\
&\quad\ \mathbb{E}[V|X_0 = i, X_1 = i-1]\Pr\{X_1 = i-1|X_0 = i\} \\
&= \frac{1}{\lambda+\mu}\frac{\lambda}{\lambda+\mu} + \left(\frac{1}{\lambda+\mu} + \frac{1}{\lambda}\right)\frac{\mu}{\lambda+\mu} = \frac{1}{\lambda}
\end{aligned}$$

Since this is true for any choice of $i > 0$, and it was assumed that $X_0 = i$, for $i > 0$, $\mathbb{E}[V] = 1/\lambda$.

<u>Exercise 6.2:</u>

The transition diagram for the embedded chain is:

$$\begin{tikzcd}
\boxed{0} \arrow[r, bend left, "1"] & \boxed{1} \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & \boxed{2} \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & \boxed{3} \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & \boxed{4} \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & \cdots \arrow[l, bend left, "3/5"]
\end{tikzcd}$$

a) The steady state probabilities satisfy $\pi_0 = \frac{3}{5}\pi_1$, $\frac{2}{5}\pi_{i-1} = \frac{3}{5}\pi_i$ for $i \geq 2$. Iterating on these equations,

$$\pi_i = \frac{2}{3}\pi_{i-1} = \left(\frac{2}{3}\right)^{i-1}\pi_1 = \frac{5}{3}\left(\frac{2}{3}\right)^{i-1}\pi_0, \text{ for } i \geq 1$$

$$1 = \sum_{i\geq 0}\pi_i = \pi_0\left[1 + \sum_{i\geq 1}\frac{5}{3}\left(\frac{2}{3}\right)^{i-1}\right] = 6\pi_0$$

Thus,

$$\begin{aligned}
\pi_0 &= \frac{1}{6} \\
\pi_i &= \frac{5}{12}\left(\frac{2}{3}\right)^i, \text{ for } i \geq 1.
\end{aligned}$$

b) The transition rates are $q_{ij} = P_{ij}v_i = P_{ij}2^i$. Therefore, $q_{01} = 1$, $q_{i,i+1} = (2/5)2^i$ and $q_{i,i-1} = (3/5)2^i$ for $i = 1, 2, ...$The steady state probabilities p_i for the Markov process are proportional to $\pi_i/v_i = (5/12)(1/3)^i$ for $i \geq 1$ and $\pi_0/v_0 = 1/6$. Normalizing so that p_i's sum to one,we get:

$$\begin{aligned} p_0 &= \frac{4}{9} \\ p_i &= \frac{10}{9}\left(\frac{1}{3}\right)^i, \text{ for } i \geq 1 \end{aligned}$$

The p_i's decay faster than π_i's. This is because the transition rate v_i increases with i. So, the mean time until a transition from the current state decreases with i.

c) Since v_i is growing unboundedly with increasing i and $P_{i,i+1}$ and $P_{i,i-1}$ is constant for all $i > 0$, $q_{ij} = P_{ij}v_i$ is also growing unboundedly with i for $j = i+1$ and $j = i-1$. Thus for any $\delta > 0$, and for large enough n, the transition probabilities of the sample Markov process will be greater than 1 (i.e., $2/5 \times 2^n\delta > 1$) which is unacceptable as a transition probability.

The embedded Markov chain of this Process is:

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & 2 \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & \cdots \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & m-1 \arrow[l, bend left, "3/5"] \arrow[r, bend left, "2/5"] & m \arrow[l, bend left, "1"] \end{tikzcd}$$

The steady state probabilities satisfy $\pi_0 = \frac{3}{5}\pi_1$, $\frac{2}{5}\pi_{i-1} = \frac{3}{5}\pi_i$ for $2 \leq i \leq m-1$ and $\frac{2}{5}\pi_{m-1} = \pi_m$. Iterating on these equations,

$$\pi_i = \frac{2}{3}\pi_{i-1} = \left(\frac{2}{3}\right)^{i-1}\pi_1 = \frac{5}{3}\left(\frac{2}{3}\right)^{i-1}\pi_0, \text{ for } 1 \leq i \leq m-1$$

$$\pi_m = \left(\frac{2}{3}\right)^{m-1}\pi_0$$

$$1 = \sum_{i=0}^{m}\pi_i = \pi_0\left[1 + \sum_{i=1}^{m-1}\frac{5}{3}\left(\frac{2}{3}\right)^{i-1} + \left(\frac{2}{3}\right)^{m-1}\right] = \pi_0\left[6 - 4\left(\frac{2}{3}\right)^{m-1}\right]$$

Thus, we will have:

$$\begin{aligned} \pi_0 &= \left[6 - 4\left(\frac{2}{3}\right)^{m-1}\right]^{-1} \\ \pi_i &= \frac{5}{3}\left(\frac{2}{3}\right)^{i-1}\left[6 - 4\left(\frac{2}{3}\right)^{m-1}\right]^{-1} = \frac{5}{2}\left(\frac{2}{3}\right)^{i}\left[6 - 4\left(\frac{2}{3}\right)^{m-1}\right]^{-1}, \text{ for } 1 \leq i \leq m-1 \\ \pi_m &= \left(\frac{2}{3}\right)^{m-1}\left[6 - 4\left(\frac{2}{3}\right)^{m-1}\right]^{-1} \end{aligned}$$

And as $m \to \infty$, this will be the same steady state distribution found in part (a).

The steady state probabilities for the sampled time are proportional to π_i/v_i. Normalizing these values give:

$$
\begin{aligned}
\sum_{i=0}^{m} \pi_i/v_i &= \left[1+\sum_{i=1}^{m-1}\frac{5}{2}\left(\frac{1}{3}\right)^{i}+\frac{1}{2}\left(\frac{1}{3}\right)^{m-1}\right]\left[6-4\left(\frac{2}{3}\right)^{m-1}\right]^{-1} \\
&= \frac{9}{4}\left(1-\frac{1}{3^m}\right)\left[6-4\left(\frac{2}{3}\right)^{m-1}\right]^{-1}
\end{aligned}
$$

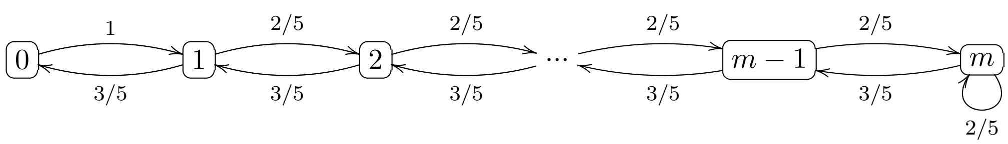
And the steady state distribution of the sampled time Markov chain will be:

$$
\begin{aligned}
p_0 &= \frac{4}{9}\left(1-\frac{1}{3^m}\right)^{-1} \\
p_i &= \frac{10}{9}\left(\frac{1}{3}\right)^{i}\left(1-\frac{1}{3^m}\right)^{-1}, \text{ for } 1 \le i \le m-1 \\
p_m &= 2\left(\frac{1}{3}\right)^{m+1}\left(1-\frac{1}{3^m}\right)^{-1}
\end{aligned}
$$

c) You observe that as $m \to \infty$, these are the same sampled time steady state distribution as found in part (b).

$$
\begin{aligned}
\lim_{m\to\infty} p_0 &= \frac{4}{9} \\
\lim_{m\to\infty} p_i &= \frac{10}{9}\left(\frac{1}{3}\right)^{i}, \text{ for } 1 \le i \le m-1 \\
\lim_{m\to\infty} p_m &= 0
\end{aligned}
$$

One could also use a truncated chain in which state m has a self transition of probability 2/5. This would change π_m to $(5/3)(2/3)^{m-1}\pi_0$ but would change the solution in a fairly negligible fashion.



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