6.254: Game Theory February 11, 2010

Lecture 4: Correlated Rationalizability

Lecturer: Asu Ozdaglar

1 Correlated Rationalizability

In this note, we allow a player to believe that the other players' actions are correlated— in other words, the other players might be in a coalition and thus pick their strategies together instead of individually. To capture this idea, we slightly modify our definition of a belief.

Definition 1.1 *A* belief *of player i about the other players' actions is a probability measure over the set S_{-i}, which we denote as $\Delta(S_{-i})$.*

Note that we allow correlation in our belief. Recall, $\Delta(S)$ denotes a probability distribution over S. One possible type of distribution is the product distribution $S_1 \times S_2 \times \ldots \times S_I$, which denotes independent mixing between the I players. In general however, the distribution $\Delta(S)$ allows correlation in the strategies of players.

Definition 1.2 *An action $s_i \in S_i$ is a rational action if there exists a belief $\alpha_{-i} \in \Delta(S_{-i})$ such that s_i is a best response to α_{-i}.*

To define rationalizable strategies, we eliminate actions that are not best responses to any belief, or in other words, we eliminate actions that are *never-best responses.* Let us next recall the definitions of "never-best response" strategy and "strictly dominated" strategy.

Definition 1.3

1. *An action s_i is a* never-best response *if for all beliefs α_{-i} there exists $\sigma_i \in \Sigma_i$ such that $u_i(\sigma_i, \alpha_{-i}) > u_i(s_i, \alpha_{-i})$.*

2. *An action s_i is strictly dominated if there exists $\sigma_i \in \Sigma_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.*

It is straightforward to show that a strictly dominated action is a never-best response. Does the other direction hold? We have shown in the previous lecture that it doesn't hold if beliefs are independent mixings. In this lecture, we will show that this direction holds for correlated beliefs.

2 Strict Dominance & Correlated Rationalizability

We first formally define the process of iterative elimination of strictly dominated strategies.

Algorithm 2.1 (Strict Dominance Iteration) *Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$. For each player $i \in \mathcal{I}$ and for each $n \geq 1$, we define S_i^n as*

$$S_i^n = \{s_i \in S_i^{n-1} \mid \text{ there is no } \quad \sigma_i \in \Sigma_i^{n-1} \text{ such that}$$
$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}^{n-1}\}.$$

Independently mix over S_i^n to get Σ_i^n. Let $D_i^\infty = \cap_{n=1}^\infty S_i^n$. We refer to the set D_i^∞ as the set of strategies of player i that survive iterated strict dominance.

We next define the process of iterative elimination of never-best response strategies. Recall our notation that $\Delta(A)$ denotes the set of probability measures over the set A [implying that the set $\Delta(S_{-i})$ denotes the set of all probability measures over the set S_{-i}, including independent mixings].

Algorithm 2.2 (Correlated Rationalizability Iteration) *Start with $\tilde{S}_i^0 = S_i$. Then, for each player $i \in \mathcal{I}$ and for each $n \geq 1$,*

$$\tilde{S}_i^n = \{s_i \in \tilde{S}_i^{n-1} \mid \text{ there exists } \quad \alpha_{-i} \in \Delta(\tilde{S}_{-i}^{n-1}) \text{ such that}$$
$$u_i(s_i, \alpha_{-i}) \geq u_i(s_i', \alpha_{-i}) \quad \text{for all } s_i' \in \tilde{S}_i^{n-1}\}.$$

Let $R_i^\infty = \cap_{n=1}^\infty \tilde{S}_i^n$. We refer to the set R_i^∞ as the set of rationalizable strategies of player i.

At an intuitive level, rationalizability asks the question "what players might do", whereas strict dominance asks the question "what players won't do, and what they won't do conditional on other players not doing certain things, etc.". Note that both iterated strict dominance and rationalizability never eliminate any strategy played with positive probability in a Nash equilibrium. Indeed both these concepts could be quite weak. Most games, including many games with a unique Nash equilibrium are not dominance solvable.

In the next proposition, we show the equivalence of iterated strict dominance and correlated rationalizability.

Proposition 2.3 *Consider a strategic form game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ with finite number of players and finite strategy spaces S_i. Then $S_i^n = \tilde{S}_i^n$ for all n.*

Proof. We first note that s_i strictly dominated implies that s_i is a never-best response. Hence $\tilde{S}_i^n \subseteq S_i^n$ for all n. We next show that if a pure strategy is a never-best response, then it is strictly dominated. We show this by contraposition.

Assume that for some player i the pure strategy $\bar{s}_i \in S_i$ is not strictly dominated. This implies that, for all $\sigma_i \in \Sigma_i^n$, there exists some $s_{-i} \in S_{-i}^n$ such that

$$u_i(\sigma_i, s_{-i}) \leq u_i(\bar{s}_i, s_{-i}). \tag{1}$$

Consider the set S_{-i}. Letting $|S_{-i}| = d$, we can write $S_{-i} = \{(s_{-i})_1, \ldots, (s_{-i})_d\}$. We define the set

$$\bar{U}_i = \Big\{ x \in \mathbb{R}^d \mid \text{there exists some } \sigma_i \in \Sigma_i \text{ such that } x_j \leq u_i(\sigma_i, (s_{-i})_j) \text{ for all } j \in \{1, \ldots, d\} \Big\}.$$

For each mixed strategy σ_i of player i, we have a point in $\bar{U}_i$, which specifies player i's payoff for every pure strategy of its opponents.

It is straightforward to verify that the set U_i is nonempty and convex. Moreover, it follows by Eq. 1, that the vector $\bar{x} = [u_i(\bar{s}_i, (s_{-i})_j)]_{j \in \{1,\ldots,d\}}$ is not an interior point of $\bar{U}_i$. By the Supporting Hyperplane Theorem (see the Appendix), there exists some $\alpha_{-i} \in \mathbb{R}^d \neq 0$ such that

$$\alpha_{-i}^T \bar{x} \geq \alpha_{-i}^T x \qquad \text{for all } x \in \bar{U}_i.$$

Since $\alpha_{-i} \neq 0$, by appropriate normalization, we can assume that $\sum_{j=1}^d (\alpha_{-i})_j = 1$. Moreover, in view of the structure of the set $\bar{U}_i$, it can be seen that $\alpha_{-i} \geq 0$. Hence α_{-i} is a (correlated) belief. From the preceding relation, we have

$$\alpha_{-i}^T\Big[u_i(\bar{s}_i, s_j)\Big]_{j\in\{1,\dots,d\}} \geq \alpha_{-i}^T\Big[u_i(\sigma_i, s_j)\Big]_{j\in\{1,\dots,d\}} \qquad \text{for all } \sigma_i \in \Sigma_i,$$

which implies that

$$u_i(\bar{s}_i, \alpha_{-i}) \geq u_i(\sigma_i, \alpha_{-i}) \qquad \text{for all } \sigma_i \in \Sigma_i.$$

Hence $\bar{s}_i$ is a best response to the belief α_{-i}. This shows that $S_i^n \subseteq \tilde{S}_i^n$ for all n and completes the proof. ∎

Final Lesson:

- Let NE_i denote the set of pure strategies of player i used with positive probability in any mixed Nash equilibrium. Then, we have $NE_i \subseteq R_i^\infty \subseteq D_i^\infty$, where R_i^∞ is the set of rationalizable strategies of player i, and D_i^∞ is the set of strategies of player i that survive iterated strict dominance.

- The latter two sets are equal when beliefs are allowed to be correlated.

3 Appendix

This section presents some basic terminology and results from real and convex analysis.

3.1 Closed and Open Sets

We say that x is a *closure point of a subset* X of $\mathbb{R}^n$ if there exists a sequence $\{x^k\} \subset X$ that converges to x. The *closure* of X, denoted by $\mathrm{cl}(X)$, is the set of all closure points of X. A subset X of $\mathbb{R}^n$ is called *closed* if $X = \mathrm{cl}(X)$. It is called *open* if its complement $\{x \mid x \notin X\}$ is closed. It is called *compact* if it is closed and bounded.

A vector $x \in X$ is an *interior point* of X if there exists a neighborhood of x that is contained in X. The set of all interior points of X is called the *interior* of X, and is denoted by $\mathrm{int}(X)$. A vector $x \in \mathrm{cl}(X)$, which is not an interior point of X is said to be a *boundary point* of X. The set of all boundary points of X is called the *boundary* of X.

3.2 Convexity and Hyperplanes

A subset C of $\mathbb{R}^n$ is called *convex* if for all $x,\ y \in C$,

$$\lambda x + (1-\lambda)y \in C, \qquad \forall\ \lambda \in [0,1].$$

Intuitively, if you take two points in a convex set and draw a line between them, then all of the points on the line must also be contained in the set.

Some of the most important principles in convexity and optimization, including duality, revolve around the use of hyperplanes, i.e., $(n-1)$-dimensional affine sets. A hyperplane has the property that it divides the space into two halfspaces.

A *hyperplane* in $\mathbb{R}^n$ is a set of the form $\{x \mid a'x = b\}$, where a is nonzero vector in $\mathbb{R}^n$ and b is a scalar. If $\bar{x}$ is any vector in a hyperplane $H = \{x \mid a'x = b\}$, then we must have $a'\bar{x} = b$, so the hyperplane can be equivalently described as

$$H = \{x \mid a'x = a'\bar{x}\},$$

or

$$H = \bar{x} + \{x \mid a'x = 0\}.$$

Thus, H is an affine set that is parallel to the subspace $\{x \mid a'x = 0\}$. This subspace is orthogonal to the vector a, and consequently, a is called the *normal* vector of H.

The sets

$$\{x \mid a'x \geq b\}, \qquad \{x \mid a'x \leq b\},$$

are called the *closed halfspaces* associated with the hyperplane (also referred to as the *positive and negative halfspaces*, respectively). The sets

$$\{x \mid a'x > b\}, \qquad \{x \mid a'x < b\},$$

are called the *open halfspaces* associated with the hyperplane.

We say that two sets C_1 and C_2 are *separated by a hyperplane* $H = \{x \mid a'x = b\}$ if each lies in a different closed halfspace associated with H, i.e., if either

$$a'x_1 \leq b \leq a'x_2, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2,$$

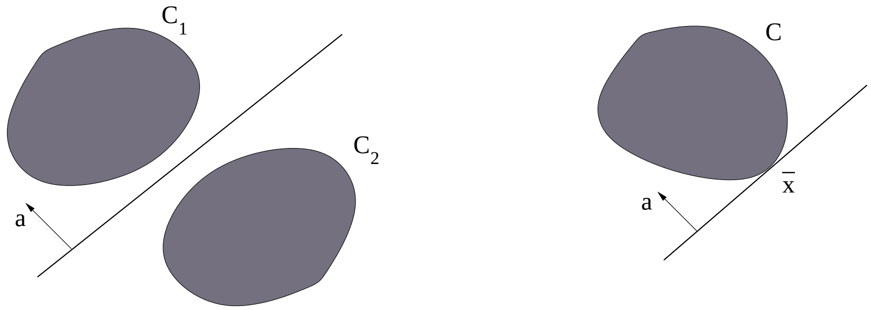

Figure 1: (a) Illustration of a hyperplane separating two sets C_1 and C_2. (b) Illustration of a hyperplane supporting a set C at a point $\bar{x}$ that belongs to the closure of C.

or

$$a'x_2 \le b \le a'x_1, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2.$$

We then also say that the hyperplane H *separates* C_1 and C_2, or that H is a *separating hyperplane* of C_1 and C_2. We use several different variants of this terminology. For example, the statement that two sets C_1 and C_2 *can be separated by a hyperplane* or that *there exists a hyperplane separating* C_1 and C_2, means that there exists a vector $a \neq 0$ such that

$$\sup_{x \in C_1} a'x \le \inf_{x \in C_2} a'x, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2;$$

(see Figure 1).

If a vector $\bar{x}$ belongs to the closure of a set C, a hyperplane that separates C and the singleton set $\{\bar{x}\}$ is said be *supporting* C *at* $\bar{x}$. Thus the statement that *there exists a supporting hyperplane of* C *at* $\bar{x}$ means that there exists a vector $a \neq 0$ such that

$$a'\bar{x} \le a'x, \qquad \forall\ x \in C,$$

or equivalently, since $\bar{x}$ is a closure point of C,

$$a'\bar{x} = \inf_{x \in C} a'x.$$

As illustrated in Figure 1, a supporting hyperplane is a hyperplane that "just touches" the set C.

We will next prove results regarding the existence of hyperplanes that separate two convex sets. The following proposition deals with the basic case where one of the two sets consists

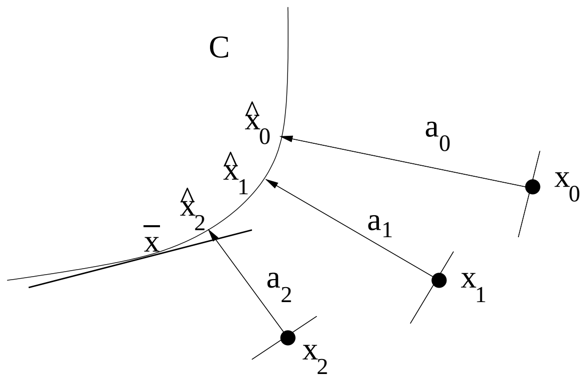

Figure 2: Illustration of the proof of the Supporting Hyperplane Theorem for the case where the vector $\bar{x}$ belongs to cl(C), the closure of C. We choose a sequence $\{x_k\}$ of vectors that do not belong to cl(C), with $x_k \to \bar{x}$, and we project x_k on cl(C). We then consider, for each k, the hyperplane that is orthogonal to the line segment connecting x_k and its projection $\hat{x}_k$, and passes through x_k. These hyperplanes "converge" to a hyperplane that supports C at $\bar{x}$.

of a single vector. The proof is based on the classical Projection Theorem (see Proposition 2.2.1 in Bertsekas, Nedic, and Ozdaglar [1]) and is illustrated in Figure 2.

Proposition 3.1 *(Supporting Hyperplane Theorem) Let C be a nonempty convex subset of $\mathbb{R}^n$ and let $\bar{x}$ be a vector in $\mathbb{R}^n$. If $\bar{x}$ is not an interior point of C, there exists a hyperplane that passes through $\bar{x}$ and contains C in one of its closed halfspaces, i.e., there exists a vector $a \neq 0$ such that*

$$a'\bar{x} \leq a'x, \qquad \forall\, x \in C.$$

Proof: Consider cl(C), the closure of C, which is a convex set. Let $\{x_k\}$ be a sequence of vectors such that $x_k \to \bar{x}$ and $x_k \notin \mathrm{cl}(C)$ for all k; such a sequence exists because $\bar{x}$ does not belong to the interior of C. If $\hat{x}_k$ is the projection of x_k on cl(C), we have (see the Projection Theorem, Proposition 2.2.1 in Bertsekas, Nedic, and Ozdaglar [1])

$$(\hat{x}_k - x_k)'(x - \hat{x}_k) \geq 0, \qquad \forall\, x \in \mathrm{cl}(C).$$

Hence we obtain for all $x \in \mathrm{cl}(C)$ and all k,

$$(\hat{x}_k - x_k)'x \geq (\hat{x}_k - x_k)'\hat{x}_k = (\hat{x}_k - x_k)'(\hat{x}_k - x_k) + (\hat{x}_k - x_k)'x_k \geq (\hat{x}_k - x_k)'x_k.$$

We can write this inequality as

$$a_k'x \geq a_k'x_k, \qquad \forall\, x \in \mathrm{cl}(C),\ \forall\, k, \tag{2}$$

where

$$a_k = \frac{\hat{x}_k - x_k}{\|\hat{x}_k - x_k\|}.$$

We have $\|a_k\| = 1$ for all k, and hence the sequence $\{a_k\}$ has a subsequence that converges to a nonzero limit a. By considering Eq. (2) for all a_k belonging to this subsequence and by taking the limit as $k \to \infty$, we obtain the desired result. ∎

Note that if $\bar{x}$ is a closure point of C, then the hyperplane of the preceding proposition supports C at $\bar{x}$. Note also that if C has empty interior, then any vector $\bar{x}$ can be separated from C as in the proposition.

Proposition 3.2 *(Separating Hyperplane Theorem) Let C_1 and C_2 be two nonempty convex subsets of $\mathbb{R}^n$. If C_1 and C_2 are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that*

$$a'x_1 \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2. \tag{3}$$

Proof: Consider the convex set

$$C = C_2 - C_1 = \{x \mid x = x_2 - x_1,\, x_1 \in C_1,\, x_2 \in C_2\}.$$

Since C_1 and C_2 are disjoint, the origin does not belong to C, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\, x \in C,$$

which is equivalent to Eq. (3). ∎

References

[1] Bertsekas D. P., Nedic A., and Ozdaglar A., Convex Analysis and Optimization, Athena Scientific, Belmont, MA, 2003.

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