6.253: Convex Analysis and Optimization
Homework 3

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Problem 1

(a) Show that a nonpolyhedral closed convex cone need not be retractive, by using as an example the cone $C = \{(u, v, w) \mid \|(u, v)\| \leq w\}$, the recession direction $d = (1, 0, 1)$, and the corresponding asymptotic sequence $\{(k, \sqrt{k}, \sqrt{k^2 + k})\}$. (This is the, so-called, second order cone, which plays an important role in conic programming; see Chapter 5.)
(b) Verify that the cone C of part (a) can be written as the intersection of an infinite number of closed halfspaces, thereby showing that a nested set sequence obtained by intersection of an infinite number of retractive nested set sequences need not be retractive.

Problem 2

Let C be a nonempty convex set in $\mathbf{R}^n$, and let M be a nonempty affine set in $\mathbf{R}^n$. Show that $M \cap rin(C) = \emptyset$ is a necessary and sufficient condition for the existence of a hyperplane H containing M, and such that $rin(C)$ is contained in one of the open halfspaces associated with H.

Problem 3

Let C_1 and C_2 be nonempty convex subsets of $\mathbf{R}^n$, and let B denote the unit ball in $\mathbf{R}^n$, $B = \{x \mid \|x\| \leq 1\}$. A hyperplane H is said to *separate strongly* C_1 and C_2 if there exists an $\epsilon > 0$ such that $C_1 + \epsilon B$ is contained in one of the open halfspaces associated with H and $C_2 + \epsilon B$ is contained in the other. Show that:
(a) The following three conditions are equivalent.
  (i) There exists a hyperplane separating strongly C_1 and C_2.
  (ii) There exists a vector $\alpha \in \mathbf{R}^n$ such that $\inf_{x \in C_1} \alpha' x > \sup_{x \in C_2} \alpha' x$.
  (iii) $\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0$, i.e., $0 \notin cl(C_2 - C_1)$.
(b) If C_1 and C_2 are disjoint, any one of the five conditions for strict separation, given in Prop. 1.5.3, implies that C_1 and C_2 can be strongly separated.

Problem 4

We say that a function $f : \mathbf{R}^n \mapsto (-\infty, \infty]$ is *quasiconvex* if all its level sets

$$V_\gamma = \{x \mid f(x) \leq \gamma\}$$

are convex. Let X be a convex subset of $\mathbf{R}^n$, let f be a quasiconvex function such that $X \cap dom(f) \neq \emptyset$, and denote $f^* = \inf_{x \in X} f(x)$.

(a) Assume that f is not constant on any line segment of X, i.e., we do not have $f(x) = c$ for some scalar c and all x in the line segment connecting any two distinct points of X. Show that every local minimum of f over X is also global.

(b) Assume that X is closed, and f is closed and proper. Let Γ be the set of all $\gamma > f^*$, and denote

$$R_f = \cap_{\gamma \in \Gamma} R_\gamma, \qquad L_f = \cap_{\gamma \in \Gamma} L_\gamma,$$

where R_γ and L_γ are the recession cone and the lineality space of V_γ, respectively. Use the line of proof of Prop. 3.2.4 to show that f attains a minimum over X if any one of the following conditions holds:

(1) $R_X \cap R_f = L_X \cap L_f$.

(2) $R_X \cap R_f \subset L_f$, and X is a polyhedral set.

Problem 5

Let $F : \mathbf{R}^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function of two vectors $x \in \mathbf{R}^n$ and $z \in \mathbf{R}^m$, and let

$$X = \left\{ x \mid \inf_{z \in \mathbf{R}^m} F(x, z) < \infty \right\}.$$

Assume that the function $F(x, \cdot)$ is closed for each $x \in X$. Show that:

(a) If for some $\bar{x} \in X$, the minimum of $F(\bar{x}, \cdot)$ over $\mathbf{R}^m$ is attained at a nonempty and compact set, the same is true for all $x \in X$.

(b) If the functions $F(x, \cdot)$ are differentiable for all $x \in X$, they have the same asymptotic slopes along all directions, i.e., for each $d \in \mathbf{R}^m$, the value of $\lim_{\alpha \to \infty} \nabla_z F(x, z + \alpha d)' d$ is the same for all $x \in X$ and $z \in \mathbf{R}^m$.