Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 6 Solutions[1]

Problem 6.1

For the following statement, verify whether it is true or false (give a proof if true, give a counterexample if false):

Assume that

(a) n, m are positive integers;

(b) $f:\ \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ and $g:\ \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^m$ are continuously differentiable functions;

(c) the ODE

$$\dot{y}(t) = g(\bar{x}, y(t))$$

has a globally asymptotically stable equilibrium for every $\bar{x} \in \mathbf{R}^n$;

(d) functions $x_0:\ [0,1] \mapsto \mathbf{R}^n$ and $y_0:\ [0,1] \mapsto \mathbf{R}^m$ are continuously differentiable and satisfy

$$\dot{x}_0(t) = f(x_0(t), y_0(t)), \quad g(x_0(t), y_0(t)) = 0 \quad \forall\ t \in [0,1].$$

Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the differential equation

$$\dot{x}(t) = f(x(t), y(t)), \quad \dot{y}(t) = \epsilon^{-1} g(x(t), y(t)),$$

has a solution $x_\epsilon:\ [0,1] \mapsto \mathbf{R}^n$, $y_\epsilon:\ [0,1] \mapsto \mathbf{R}^m$ such that $x_e(0) = x_0(0)$, $y_\epsilon(0) = y_0(0)$, and $x_\epsilon(t)$ converges to $x_0(t)$ as $\epsilon \to 0$ for all $t \in [0,1]$.

[1] Version of November 12, 2003

The statement is false. To see this, consider the case when $n = m = 1$,

$$f(x,y) = 3y^2, \quad g(x,y) = x - y^3, \quad x_0(t) = t^3, \quad y_0(t) = t.$$

Then conditions (a)-(d) are satisfied. However, every solution x_e, y_e of the singularly perturbed equation with initial conditions $x_e(0) = y_e(0) = 0$ will be identically equal to zero. Hence, the convergence conclusion does not hold.

Problem 6.2

FIND ALL VALUES OF PARAMETER $k \in \mathbf{R}$ FOR WHICH SOLUTIONS OF THE ODE SYSTEM

$$\begin{aligned} \dot{x}_1 &= -x_1 + k(x_1^3 - 3x_1x_2^2) \\ \dot{x}_2 &= -x_2 + k(3x_1^2x_2 - x_2^3) \end{aligned}$$

WITH ALMOST ALL INITIAL CONDITIONS CONVERGE TO ZERO AS $t \to \infty$. **Hint:** USE WEIGHTED VOLUME MONOTONICITY WITH DENSITY FUNCTION

$$\rho(x) = |x|^\gamma.$$

ALSO, FOR THOSE WHO REMEMBER COMPLEX ANALYSIS, IT IS USEFUL TO PAY ATTENTION TO THE FACT THAT

$$(x_1^3 - 3x_1x_2^2) + j(3x_1^2x_2 - x_2^3) = (x_1 + jx_2)^3.$$

Using Theorem 11.5 with $\rho(x) = |x|^{-6}$ yields convergence of almost every solution to zero for *all* $k \in \mathbf{R}$. Indeed, for

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + k(x_1^3 - 3x_1x_2^2) \\ -x_2 + k(3x_1^2x_2 - x_2^3) \end{bmatrix}$$

we have

$$\operatorname{div}(\rho(x)f(x)) = 4|x|^{-6} \quad \text{for} \quad x \neq 0.$$

In addition, $|f(x)|$ grows not faster than $|x|^3$ as $|x| \to \infty$, and hence $|f(x)|\rho(x)$ is integrable over the set $|x| \geq 1$. Since ρ is positive, the assumptions of Theorem 11.5 are satisfied.

Problem 6.3

EQUATIONS FOR STEERING A TWO-WHEELED VEHICLE (WITH ONE WHEEL USED FOR STEERING AND THE OTHER WHEEL FIXED) ON A PLANAR SURFACE ARE GIVEN BY

$$\begin{aligned} \dot{x}_1 &= \cos(x_3)u_1, \\ \dot{x}_2 &= \sin(x_3)u_1, \\ \dot{x}_3 &= x_4u_1, \\ \dot{x}_4 &= u_2, \end{aligned}$$

WHERE x_1, x_2 ARE DECART COORDINATES OF THE FIXED WHEEL, x_3 IS THE ANGLE BETWEEN THE VEHICLE'S AXIS AND THE x_1 AXIS, x_4 IS A PARAMETER CHARACTERIZING THE STEERING ANGLE, u_1, u_2 ARE FUNCTIONS OF TIME WITH VALUES RESTRICTED TO THE INTERVAL $[-1, 1]$.

(a) FIND ALL STATES REACHABLE (TIME FINITE BUT NOT FIXED) FROM A GIVEN INITIAL STATE $\bar{x}_0 \in \mathbf{R}^4$ BY USING APPROPRIATE CONTROL.

This is a driftless system of the form

$$\dot{x}(t) = g_1(x(t))u_1(t) + g_2(x(t))u_2(t),$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbf{R}^4, \quad g_1(x) = \begin{bmatrix} \cos(x_3) \\ \sin(x_3) \\ x_4 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$g_3(x) = [g_1, g_2](x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_4(x) = [g_1, g_3](x) = \begin{bmatrix} -\sin(x_3) \\ \cos(x_3) \\ 0 \\ 0 \end{bmatrix}.$$

Since vectors $g_1(x), g_2(x), g_3(x), g_4(x)$ form a basis in $\mathbf{R}^4$ for all $x \in \mathbf{R}^4$, every state is reachable from every other state in arbitrary (positive) time, provided that sufficiently large piecewise constant control values can be used. Since in our case the control values have limited range, every state is reachable from every other state in sufficient time.

(b) DESIGN AND TEST IN COMPUTER SIMULATION AN ALGORITHM FOR MOVING IN FINITE TIME THE STATE FROM A GIVEN INITIAL POSITION $\bar{x}_0 \in \mathbf{R}^4$ TO AN ARBITRARY REACHABLE STATE $\bar{x}_1 \in \mathbf{R}^4$.

A very straightforward control algorithm for controlling the system is based on using commutators of differential flows of vector fields as approximations for the flows defined by Lie brackets of these fields. Let $S_k^t(x)$ denote the differential flow defined by g_k. While the state transitions S_1 and S_2 can be implemented directly by applying vector fields g_1 and g_2, we will approximate S_3^t by $S_1^{-t_1} \circ S_2^{-t_2} \circ S_1^{t_1} \circ S_2^{t_2}$, where $t_1 = \pm\sqrt{t}$ and $t_2 = \sqrt{t}$. Similarly, S_4 will be implemented by composing S_1 and the approximations of S_3 described before. Now, for a given current state vector $x(t)$, and the desired terminal state x_1, select the vector field g_k for which $v = g_k(x(t))'(x_1 - x(t))/|g_k(x(t))|^2$ is maximal, and then try to implement moving with the flow S_k for some time proportional to v.

A MATLAB simulation of the algorithm is provided in `hw6_3_6243_2003.m`