Lectures on Dynamic Systems and Control

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Chapter 11

Continuous-Time Linear State-Space Models

11.1 Introduction

In this chapter, we focus on the solution of CT state-space models. The development here follow the previous chapter.

11.2 The Time-Varying Case

Consider the nth-order continuous-time linear state-space description

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \ . \end{aligned} \tag{11.1}$$

We shall always assume that the coefficient matrices in the above model are sufficiently well behaved for there to *exist* a *unique* solution to the state-space model for any specified initial condition $x(t_0)$ and any integrable input $u(t)$. For instance, if these coefficient matrices are piecewise continuous, with a finite number of discontinuities in any finite interval, then the desired existence and uniqueness properties hold.

We can describe the solution of (11.1) in terms of a matrix function $\Phi(t,\tau)$ that has the following two properties:

$$\dot{\Phi}(t,\tau) = A(t)\Phi(t,\tau) \ , \tag{11.2}$$

$$\Phi(\tau,\tau) = I \ . \tag{11.3}$$

This matrix function is referred to as the **state transition matrix**, and under our assumption on the nature of $A(t)$ it turns out that the state transition matrix *exists* and is *unique*.

We will show that, given $x(t_0)$ and $u(t)$,

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \ . \tag{11.4}$$

Observe again that, as in the DT case, the terms corresponding to the zero-input and zero-state responses are evident in (11.4). In order to verify (11.4), we differentiate it with respect to t:

$$\dot{x}(t) = \dot{\Phi}(t, t_0)x(t_0) + \int_{t_0}^{t} \dot{\Phi}(t, \tau)B(\tau)u(\tau)d\tau + \Phi(t, t)B(t)u(t) \ . \tag{11.5}$$

Using (11.2) and (11.3),

$$\dot{x}(t) = A(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} A(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + B(t)u(t) \ . \tag{11.6}$$

Now, since the integral is taken with respect to τ, $A(t)$ can be factored out:

$$\begin{aligned}
\dot{x}(t) &= A(t)\left[\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau\right] + B(t)u(t) \ , && (11.7) \\
&= A(t)x(t) + B(t)u(t) \ , && (11.8)
\end{aligned}$$

so the expression in (11.4) does indeed satisfy the state evolution equation. To verify that it also matches the specified initial condition, note that

$$x(t_0) = \Phi(t_0, t_0)x(t_0) = x(t_0). \tag{11.9}$$

We have now shown that the matrix function $\Phi(t, \tau)$ satisfying (11.2) and (11.3) yields the solution to the continuous-time system equation (11.1).

Exercise: Show that $\Phi(t, \tau)$ must be nonsingular. (Hint: Invoke our claim about uniqueness of solutions.)

The question that remains is how to find the state transition matrix. For a general linear time-varying system, there is no analytical expression that expresses $\Phi(t, \tau)$ analytically as a function of $A(t)$. Instead, we are essentially limited to numerical solution of the equation (11.2) with the boundary condition (11.3). This equation may be solved one column at a time, as follows. We numerically compute the respective solutions $x^i(t)$ of the homogeneous equation

$$\dot{x}(t) = A(t)x(t) \tag{11.10}$$

for each of the n initial conditions below:

$$x^1(t_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x^2(t_0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad x^n(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} .$$

Then

$$\Phi(t, t_0) = \left[\begin{array}{ccc} x^1(t) & \ldots & x^n(t) \end{array} \right]. \tag{11.11}$$

In summary, knowing n solutions of the homogeneous system for n independent initial conditions, we are able to construct the general solution of this linear time varying system. The underlying reason this construction works is that solutions of a linear system may be superposed, and our system is of order n.

Example 11.1 A Special Case

Consider the following time-varying system

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha(t) & \beta(t) \\ -\beta(t) & \alpha(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where $\alpha(t)$ and $\beta(t)$ are continuous functions of t. It turns out that the special structure of the matrix $A(t)$ here permits an analytical solution. Specifically, verify that the state transition matrix of the system is

$$\Phi(t, t_0) = \begin{bmatrix} \exp(\int_{t_0}^t \alpha(\tau)d\tau)\cos(\int_{t_0}^t \beta(\tau)d\tau) & \exp(\int_{t_0}^t \alpha(\tau)d\tau)\sin(\int_{t_0}^t \beta(\tau)d\tau) \\ -\exp(\int_{t_0}^t \alpha(\tau)d\tau)\sin(\int_{t_0}^t \beta(\tau)d\tau) & \exp(\int_{t_0}^t \alpha(\tau)d\tau)\cos(\int_{t_0}^t \beta(\tau)d\tau) \end{bmatrix}$$

The secret to solving the above system — or equivalently, to obtaining its state transition matrix — is to transform it to polar co-ordinates via the definitions

$$\begin{aligned} r^2(t) &= (x_1)^2(t) + (x_2)^2(t) \\ \theta(t) &= \tan^{-1}\left(\frac{x_2}{x_1}\right) . \end{aligned}$$

We leave you to deduce now that

$$\begin{aligned} \frac{d}{dt} r^2 &= 2\alpha r^2 \\ \frac{d}{dt}\theta &= -\beta . \end{aligned}$$

The solution of this system of equations is then given by

$$r^2(t) = \exp\left(2 \int_{t_0}^t \alpha(\tau)d\tau\right) r^2(t_0)$$

and

$$\theta(t) = \theta(t_0) - \int_{t_0}^t \beta(\tau)d\tau$$

Further Properties of the State Transition Matrix

The first property that we present involves the composition of the state transition matrix evaluated over different intervals. Suppose that at an arbitrary time t_0 the state vector is $x(t_0) = x_0$, with x_0 being an arbitrary vector. In the absence of an input the state vector at time t is given by $x(t) = \Phi(t, t_0)x_0$. At any other time t_1, the state vector is given by $x(t_1) = \Phi(t_1, t_0)x_0$. We can also write

$$\begin{aligned} x(t) &= \Phi(t, t_1)x(t_1) = \Phi(t, t_1)\Phi(t_1, t_0)x_0 \\ &= \Phi(t, t_0)x_0. \end{aligned}$$

Since x_0 is arbitrary, it follows that

$$\Phi(t, t_1)\Phi(t_1, t_0) = \Phi(t, t_0)$$

for any t_0 and t_1. (Note that since the state transition matrix in CT is alway invertible, there is no restriction that t_1 lie between t_0 and t — unlike in the DT case, where the state transition matrix may not be invertible).

Another property of interest (but one whose derivation can be safely skipped on a first reading) involves the determinant of the state transition matrix. We will now show that

$$\det(\Phi(t, t_0)) = \exp\Big(\int_{t_0}^{t} \text{trace}[A(\tau)]d\tau\Big) \ , \tag{11.12}$$

a result known as the *Jacobi-Liouville* formula. Before we derive this important formula, we need the following fact from matrix theory. For an $n \times n$ matrix M and a real parameter ϵ, we have

$$\det(I + \epsilon M) = 1 + \epsilon \, \text{trace}\,(M) + O(\epsilon^2) \ ,$$

where $O(\epsilon^2)$ denotes the terms of order greater than or equal to ϵ^2. In order to verify this fact, let U be a similarity transformation that brings M to an upper triangular matrix T, so $M = U^{-1}TU$. Such a U can always be found, in many ways. (One way, for a diagonalizable matrix, is to pick U to be the modal matrix of M, in which case T is actually diagonal; there is a natural extension of this approach in the non-diagonalizable case.) Then the eigenvalues $\{\lambda_i\}$ of M and T are identical, because similarity transformations do not change eigenvalues, and these numbers are precisely the diagonal elements of T. Hence

$$\begin{aligned} \det(I + \epsilon M) &= \det(I + \epsilon T) \\ &= \Pi_{i=1}^{n} \, (1 + \epsilon\lambda_i) \\ &= 1 + \epsilon \, \text{trace}\,(M) + O(\epsilon^2). \end{aligned}$$

Returning to the proof of (11.12), first observe that

$$\begin{aligned} \Phi(t + \epsilon, t_0) &= \Phi(t, t_0) + \epsilon \frac{d}{dt}\Phi(t, t_0) + O(\epsilon^2) \\ &= \Phi(t, t_0) + \epsilon A(t)\Phi(t, t_0) + O(\epsilon^2). \end{aligned}$$

The derivative of the determinant of $\Phi(t, t_0)$ is given by

$$\begin{aligned}
\frac{d}{dt}\det[\Phi(t,t_0)] &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\det[\Phi(t+\epsilon,t_0)] - \det[\Phi(t,t_0)]\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\det[\Phi(t,t_0) + \epsilon A(t)\Phi(t,t_0)] - \det[\Phi(t,t_0)]\Big) \\
&= \det(\Phi(t,t_0))\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\det[I+\epsilon A(t)] - 1\right) \\
&= \text{trace}\,[A(t)]\det[\Phi(t,t_0)].
\end{aligned}$$

Integrating the above equation yields the desired result, (11.12).

11.3 The LTI Case

For linear time-invariant systems in continuous time, it is possible to give an explicit formula for the state transition matrix, $\Phi(t, \tau)$. In this case $A(t) = A$, a constant matrix. Let us *define* the **matrix exponential** of A by an infinite series of the same form that is (or may be) used to define the scalar exponential:

$$\begin{aligned}
e^{(t-t_0)A} &= I + (t-t_0)A + \frac{1}{2!}(t-t_0)^2A^2 + \ldots \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}(t-t_0)^kA^k. \qquad (11.13)
\end{aligned}$$

It turns out that this series is as nicely behaved as in the scalar case: it converges absolutely for all $A \in \mathbb{R}^{n\times n}$ and for all $t \in \mathbb{R}$, and it can be differentiated or integrated term by term. There exist methods for computing it, although the task is fraught with numerical difficulties.

With the above definition, it is easy to verify that the matrix exponential satisfies the defining conditions (11.2) and (11.3) for the state transition matrix. The solution of (11.1) in the LTI case is therefore given by

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \qquad (11.14)$$

After determining $x(t)$, the system output can be obtained by

$$y(t) = Cx(t) + Du(t). \qquad (11.15)$$

Transform-Domain Solution of LTI Models

We can now parallel our transform-domain treatment of the DT case, except that now we use the one-sided Laplace transform instead of the $\mathcal{Z}$-transform:

Definition 11.1 *The one-sided Laplace transform, $F(s)$, of the signal $f(t)$ is given by*

$$F(s) = \int_{t=0-}^{\infty} e^{-st} f(t)\, dt$$

for all s where the integral is defined, denoted by the region of convergence (R.O.C.).

The various properties of the Laplace transform follow. The shift property of $\mathcal{Z}$ transforms that we used in the DT case is replaced by the following differentiation property: Suppose that $f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} F(s)$. Then

$$g(t) = \frac{df(t)}{dt} \implies G(s) = sF(s) - f(0-)$$

Now, given the state-space model (11.1) in the LTI case, we can take transforms on both sides of the equations there. Using the transform property just described, we get

$$\begin{aligned} sX(s) - x(0-) &= AX(s) + BU(s) && (11.16) \\ Y(s) &= CX(s) + DU(s). && (11.17) \end{aligned}$$

This is solved to yield

$$\begin{aligned} X(s) &= (sI - A)^{-1}x(0-) + (sI - A)^{-1}BU(s) \\ Y(s) &= C(sI - A)^{-1}x(0-) + \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{\text{Transfer Function}} U(s) && (11.18) \end{aligned}$$

which is very similar to the DT case.

An important fact that emerges on comparing (11.18) with its time-domain version (11.14) is that

$$\mathcal{L}\left(e^{At}\right) = (sI - A)^{-1}.$$

Therefore one way to compute the state transition matrix (a good way for small examples!) is by evaluating the entry-by-entry inverse transform of $(sI - A)^{-1}$.

Example 11.2 Find the state transition matrix associated with the (non-diagonalizable!) matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Using the above formula,

$$\begin{aligned}\mathcal{L}\left(e^{At}\right) &= (sI-A)^{-1} = \begin{bmatrix} s-1 & -2 \\ 0 & s-1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}.\end{aligned}$$

By taking the inverse Laplace transform of the above matrix we get

$$e^{At} = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}.$$

Exercises

Exercise 11.1 Companion Matrices

(a) The following two matrices and their transposes are said to be *companion matrices* of the polynomial $q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0$. Determine the characteristic polynomials of these four matrices, and hence explain the origin of the name. (Hint: First find explanations for why all four matrices must have the same characteristic polynomial, then determine the characteristic polynomial of any one of them.)

$$A_1 = \begin{pmatrix} -q_{n-1} & 1 & 0 & \ldots & 0 \\ -q_{n-2} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & 0 & 0 & \ldots & 1 \\ -q_0 & 0 & 0 & \ldots & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -q_0 & -q_1 & -q_2 & \ldots & -q_{n-1} \end{pmatrix}$$

(b) Show that the matrix A_2 above has only one (right) eigenvector for each distinct eigenvalue λ_i, and that this eigenvector is of the form $[1 \ \ \lambda_i \ \ \lambda_i^2 \ \ \ldots \ \ \lambda_i^{n-1}]^T$.

(c) If

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}$$

what are A^k and e^{At}? (Your answers may be left as a product of three — or fewer — matrices; do not bother to multiply them out.)

Exercise 11.2 Suppose you are given the state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with an input $u(t)$ that is piecewise constant over intervals of length T:

$$u(t) = u[k] \ , \qquad kT < t \leq (k+1)T$$

(a) Show that the sampled state $x[k] = x(kT)$ is governed by a *sampled-data state-space model* of the form

$$x[k+1] = Fx[k] + Gu[k]$$

for constant matrices F and G (i.e. matrices that do not depend on t or k), and determine these matrices in terms of A and B. (Hint: The result will involve the matrix exponential, e^{At}.) How are the eigenvalues and eigenvectors of F related to those of A?

(b) Compute F and G in the above discrete-time sampled-data model when

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} , \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Suppose we implement a *state feedback control law* of the form $u[k] = Hx[k]$, where H is a gain matrix. What choice of H will cause the state of the resulting closed-loop system, $x[k+1] = (F + GH)x[k]$, to go to 0 in at most two steps, from any initial condition (H is then said to produce "deadbeat" behavior)? To simplify the notation for your calculations, denote $\cos \omega_0 T$ by c and $\sin \omega_0 T$ by s. Assume now that $\omega_0 T = \pi/6$, and *check your result* by substituting in your computed H and seeing if it does what you intended.

(d) For $\omega_0 T = \pi/6$ and $\omega_0 = 1$, your matrices from (b) should work out to be

$$F = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} , \quad G = \begin{pmatrix} 1 - (\sqrt{3}/2) \\ 1/2 \end{pmatrix}$$

Use Matlab to compute and plot the response of each of the state variables from $k = 0$ to $k = 10$, assuming $x[0] = [4\,,\,0]^T$ and with the following choices for $u[k]$:

- (i) the open-loop system, with $u[k] = 0$;
- (ii) the closed-loop system with $u[k] = Hx[k]$, where H is the feedback gain you computed in (c), with $\omega_0 = 1$; also plot $u[k]$ in this case.

(e) Now suppose the controller is computer-based. The above control law $u[k] = Hx[k]$ is implementable if the time taken to compute $Hx[k]$ is negligible compared to T. Often, however, it takes a considerable fraction of the sampling interval to do this computation, so the control that is applied to the system at time k is forced to use the state measurement at the previous instant. Suppose therefore that $u[k] = Hx[k-1]$. Find a state-space model for the closed-loop system in this case, written in terms of F, G, and H. (Hint: The computer-based controller now has memory!) What are the eigenvalues of the closed-loop system now, with H as in (c)? Again use Matlab to plot the response of the system to the same initial condition as in (d), and compare with the results in (d)(ii). Is there another choice of H that could yield deadbeat behavior? If so, find it; if not, suggest how to modify the control law to obtain deadbeat behavior.

Exercise 11.3 Given the matrix

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix},$$

show that

$$\exp\left(t \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}\right) = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}$$

Exercise 11.4 Suppose A and B are constant square matrices. Show that

$$\exp\left(t \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \begin{bmatrix} e^{tA} & 0 \\ 0 & e^{tB} \end{bmatrix}.$$

Exercise 11.5 Suppose A and B are constant square matrices. Show that the solution of the following system of differential equations,

$$\dot{x}(t) = e^{-tA} B e^{tA} x(t) \ ,$$

is given by

$$x(t) = e^{-tA} e^{(t-t_0)(A+B)} e^{t_0 A} x(t_0) \ .$$

Exercise 11.6 Suppose A is a constant square matrix, and $f(t)$ is a continuous scalar function of t. Show that the state transition matrix for the system

$$\dot{x}(t) = f(t) A \, x(t)$$

is given by

$$\Phi(t, t_0) = \exp\left((\int_{t_0}^{t} f(\tau) d\tau) A \right) .$$

Exercise 11.7 (*Floquet Theory*). Consider the system

$$\dot{x}(t) = A(t) x(t)$$

where $A(t)$ is a periodic matrix with period T, so $A(t+T) = A(t)$. We want to study the state transition matrix $\Phi(t, t_0)$ associated with this periodically time-varying system.

1. First let us start with the state transition matrix $\Phi(t, 0)$, which satisfies

$$\begin{aligned} \dot{\Phi} &= A(t)\Phi \\ \Phi(0,0) &= I. \end{aligned}$$

   Define the matrix $\Psi(t, 0) = \Phi(t+T, 0)$ and show that Ψ satisfies

$$\begin{aligned} \dot{\Psi}(t,0) &= A(t)\Psi(t,0) \\ \Psi(0,0) &= \Phi(T,0). \end{aligned}$$

2. Show that this implies that $\Phi(t+T, 0) = \Phi(t, 0)\Phi(T, 0)$.
3. Using Jacobi-Liouville formula, show that $\Phi(T, 0)$ is invertible and therefore can be written as $\Phi(T, 0) = e^{TR}$.
4. Define

$$P(t)^{-1} = \Phi(t, 0) e^{-tR},$$

   and show that $P(t)^{-1}$, and consequently $P(t)$, are periodic with period T. Also show that $P(T) = I$. This means that

$$\Phi(t, 0) = P(t)^{-1} e^{tR}.$$

5. Show that $\Phi(0, t_0) = \Phi^{-1}(t_0, 0)$. Using the fact that $\Phi(t, t_0) = \Phi(t, 0)\Phi(0, t_0)$, show that

$$\Phi(t, t_0) = P(t)^{-1} e^{(t-t_0)R} P(t_0).$$

   What is the significance of this result?

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