Solution 2

Exercise 4.1

LQ Problem with Forecasts:

$$x_{k+1} = A_k x_k + B_k u_k + w_k \qquad k = 0, 1, \ldots, N-1$$

$$\text{cost:} \qquad \underset{\substack{w_k \\ k=0,1,\ldots,N-1}}{E} \left\{ x_N' Q_N x_N + \sum_{k=0}^{N-1} (x_k' Q_k x_k + u_k' R_k u_k) \right\}$$

Let:

$$\begin{aligned} y_k &\triangleq \text{Forecast available at the beginning of period } k \\ P_{k \mid y_k} &\triangleq \text{p.d.f. of } w_k \text{ given } y_k \\ p_{y_k}^k &\triangleq \text{a priori p.d.f. of } y_k \text{ at stage } k \end{aligned}$$

Following sections 1.4 and 4.1 we have the following DP algorithm:

$$J_N(x_N, y_N) = x_N' Q_N x_N$$

$$J_k(x_k, y_k) = \min_{u_k} \underset{\substack{w_k \\ k=0,1,\ldots,N-1}}{E} \left\{ x_k' Q_k x_k + u_k' R_k u_k + \sum_{i=1}^{n} p_i^{k+1} J_{k+1}(x_{k+1}, i) \,\Big|\, y_k \right\}$$

where the expectation is taken with respect to $P_{k \mid y_k}$.

Theorem: Under the conditions of the problem:

$$J_k(x_k, y_k) = x_k' K_k x_k + x_k' b_k(y_k) + c_k(y_k) \qquad k = 0, 1, \ldots, N,$$

where $b_k(y_k)$ is an n-dimensional vector, $c_k(y_k)$ is a scalar, and K_k is generated by the discrete-time Riccati equation.

Proof: The proof will follow by induction, deriving $\mu_k^*(x_k, y_k)$ on the way. For $k = N$ the theorem is clearly true.

Assume: $J_{k+1}(x_{k+1}, y_{k+1}) = x'_{k+1}K_{k+1}x_{k+1} + x'_{k+1}b_{k+1}(y_{k+1}) + c_{k+1}(y_{k+1})$. Then:

$$
\begin{aligned}
J_k(x_k, y_k) = \min_{u_k} \mathop{E}_{w_k} \Big\{ & x'_k Q_k x_k + u'_k R_k u_k + \\
& \sum_{i=1}^{n} p_i^{k+1} \big[x'_{k+1} K_{k+1} x_{k+1} + x'_{k+1} b_{k+1}(i) + c_{k+1}(i) \big] \,\big|\, y_k \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= x'_k Q_k x_k + \min_{u_k} \Big\{ u'_k R_k u_k + \mathop{E}_{w_k} \Big\{ (A_k x_k + B_k u_k + w_k)' K_{k+1} (A_k x_k + B_k u_k + w_k) + \\
&\qquad (A_k x_k + B_k u_k + w_k)' \underbrace{\sum_{i=1}^{n} p_i^{k+1} b_{k+1}(i)}_{b_{k+1}} \,\big|\, y_k \Big\} + \underbrace{\sum_{i=1}^{n} p_i^{k+1} c_{k+1}(i)}_{\gamma_{k+1}} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= x'_k Q_k x_k + x'_k A'_k K_{k+1} A_k x_k + 2 x'_k A'_k K_{k+1} \, E\{w_k \,|\, y_k\} + \\
&\qquad E\{w'_k K_{k+1} w_k \,|\, y_k\} + x'_k A'_k b_{k+1} + b'_{k+1} \, E\{w_k \,|\, y_k\} + \gamma_{k+1} + \\
&\qquad \min_{u_k} \Big\{ u'_k (R_k + B'_k K_{k+1} B_k) u_k + 2 u'_k B'_k K_{k+1} A_k x_k + 2 u'_k B'_k K_{k+1} \, E\{w_k \,|\, y_k\} \\
&\qquad + u'_k B'_k b_{k+1} \Big\}
\end{aligned}
$$

We know that $R_k > 0$. This implies that $R_k + B'_k K_{k+1} B_k > 0$, since $K_{k+1} \geq 0$ by induction. Thus, we can find the minimum by finding the stationary point via differentiation:

$$
2(R_k + B'_k K_{k+1} B_k) u^*_k + 2 B'_k K_{k+1} (A_k x_k + E\{w_k \,|\, y_k\} + B'_k b_{k+1}) = 0
$$

This gives the optimal control law:

$$
u^*_k = \mu^*_k(x_k, y_k) = -(R_k + B'_k K_{k+1} B_k)^{-1} B'_k K_{k+1} (A_k x_k + E\{w_k \,|\, y_k\}) + \alpha_k
$$

where $\alpha_k = -\frac{1}{2}(R_k + B'_k K_{k+1} B_k)^{-1} B'_k b_{k+1}$. Now we substitute u^*_k back into $J_k(x_k, y_k)$ yielding:

$$
\begin{aligned}
& J_k(x_k, y_k) \\
&= \underbrace{x'_k (Q_k + A'_k K_{k+1} A_k) x_k - x'_k [A'_k K_{k+1} B_k (R_k + B'_k K_{k+1} B_k)^{-1} B'_k K_{k+1} A_k] x_k}_{\textit{quadratic term } x'_k K_k x_k} + \\
& \underbrace{\begin{aligned} & 2 x'_k A'_k K_{k+1} \, E\{w_k \,|\, y_k\} + x'_k A'_k b_{k+1} \\ & - 2 x'_k A'_k K_{k+1} B_k (R_k + B'_k K_{k+1} B_k)^{-1} B'_k K_{k+1} \, E\{w_k \,|\, y_k\} \\ & - \frac{1}{2} x'_k A'_k K_{k+1} B_k (R_k + B'_k K_{k+1} B_k)^{-1} B'_k b_{k+1} \end{aligned}}_{\textit{linear term } x'_k b_k(y_k)} + \\
& \underbrace{\begin{aligned} & E\{w'_k K_{k+1} w_k \,|\, y_k\} + b'_{k+1} \, E\{w_k \,|\, y_k\} + \gamma_{k+1} \\ & - E\{w'_k \,|\, y_k\} K_{k+1} B_k (R_k + B'_k K_{k+1} B_k)^{-1} B'_k K_{k+1} \, E\{w_k \,|\, y_k\} \\ & - \frac{1}{4} b'_{k+1} B_k (R_k + B'_k K_{k+1} B_k)^{-1} B'_k b_{k+1} \end{aligned}}_{\textit{constant term } c_k(y_k)}
\end{aligned}
$$

Q.E.D.

Exercise 4.2

We note first from integral tables:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}\, e^{(b^2-4ac)/4a} \quad for\ a > 0$$

Let w be a normal random variable with zero mean and variance $\sigma^2 < \frac{1}{2}$. Using this definite integral, we have:

$$\begin{aligned}
E\left\{e^{(a+w)^2}\right\} &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(a+w)^2} e^{-\frac{w^2}{2\sigma^2}} dw \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2\sigma^2}w^2 - w^2 - 2aw - a^2\right)} dw \\
&= \frac{1}{\sigma\sqrt{2\pi}} \sqrt{\frac{\pi}{\frac{1}{2\sigma^2} - 1}}\; e^{\frac{4a^2 + 4a^2(\frac{1}{2\sigma^2}-1)}{4(\frac{1}{2\sigma^2}-1)}} \\
&= \frac{1}{\sqrt{1-2\sigma^2}}\, e^{\frac{a^2}{1-2\sigma^2}}
\end{aligned}$$

Theorem: If the DP algorithm has a finite minimizing value at each step,

$$J_k(x_k) = \alpha_k\, e^{\beta_k x_k^2}, \quad \beta_k > 0,\ k = 0, 1, \ldots$$

Proof: The proof follows by induction. For $k = N$,

$$J_N(x_N) = e^{x_N^2}$$

Assume that:

$$J_{k+1}(x_{k+1}) = \alpha_{k+1}\, e^{\beta_{k+1} x_{k+1}^2}, \quad \beta_{k+1} > 0$$

Then:

$$J_k(x_k) = \min_{u_k} \left\{ e^{x_k^2 + r u_k^2} \mathop{E}_{w_k}\{J_{k+1}(x_{k+1})\} \right\}$$

Assume that the expected value in the previous expression exists. (Note that an existence proof will depend on the linearity of the system). Then:

$$\begin{aligned}
J_k(x_k) &= \min_{u_k} \left\{ e^{x_k^2 + r u_k^2}\ \alpha_{k+1} \mathop{E}_{w_k} \left\{ e^{\beta_{k+1}(a_k x_k + b_k u_k + w_k)^2} \right\} \right\} \\
&= \min_{u_k} \left\{ e^{x_k^2 + r u_k^2} \frac{\alpha_{k+1}}{\sqrt{1 - 2\beta_{k+1}\sigma^2}}\, e^{\frac{\beta_{k+1}(a_k x_k + b_k u_k)^2}{1 - 2\beta_{k+1}\sigma^2}} \right\}
\end{aligned}$$

since w_k is a Gaussian r.v. with 0 mean and $\beta_{k+1}\sigma^2$ variance. Thus, the assumption of finite minimum is equivalent to having $2\beta_{k+1}\sigma^2 < 1$ at each stage. The calculation of the minimum reduces to minimizing:

$$x_k^2 + r u_k^2 + \frac{\beta_{k+1}(a_k x_k + b_k u_k)^2}{1 - 2\beta_{k+1}\sigma^2}$$

over u_k. But this is a quadratic where the coefficient on u_k^2 is:

$$r + \frac{\beta_{k+1} b_k^2}{1 - 2\beta_{k+1}\sigma^2}$$

which is positive. Thus, we are minimizing a convex function and:

$$u_k^* = \mu_k^*(x_k) = \gamma_k x_k$$

Substituting u_k back into $J_k(x_k)$ yields:

$$\begin{aligned} J_k(x_k) &= \frac{\alpha_{k+1}}{\sqrt{1 - 2\beta_{k+1}\sigma^2}}\, e^{\left[1 + r\gamma_k^2 + \frac{\beta_{k+1}}{1 - 2\beta_{k+1}\sigma^2}(a_k + b_k\gamma_k)^2\right] x_k^2} \\ &= \alpha_k\, e^{\beta_k x_k^2}, \quad with\ \beta_k > 0 \end{aligned}$$

Q.E.D.

As an example where without the Gaussian assumption the control is not linear, consider the following pmf for w_{N-1}:

$$Pr(w_{N-1} = \xi) = \begin{cases} 1/4, & \text{if } |\xi| = 1 \\ 1/2, & \text{if } \xi = 0 \end{cases}$$

Then:

$$\begin{aligned} J_{N-1}(x_{N-1}) &= \min_{u_{N-1}} \left\{ e^{x_{N-1}^2 + r u_{N-1}^2} \underset{w_{N-1}}{E} \left\{ e^{(a_{N-1}x_{N-1} + b_{N-1}u_{N-1} + w_{N-1})^2} \right\} \right\} \\ &= \min_{u_{N-1}} \left\{ \frac{1}{2}\, e^{x_{N-1}^2 + r u_{N-1}^2} \left[e^{(a_{N-1}x_{N-1} + b_{N-1}u_{N-1} + 1)^2} + e^{(a_{N-1}x_{N-1} + b_{N-1}u_{N-1} - 1)^2} \right] \right\} \end{aligned}$$

Then, in general, $u_{N-1}^* \neq \gamma_{N-1} x_{N-1}$.

Exercise 4.5

a) Following the analysis in Section 4.2, we have that the DP algorithm is

$$J_N(x_N) = r_N(x_N),$$

$$J_k(x_k) = r_k(x_k) + \min_{u_k \geq 0} \left[E\left\{ J_{k+1}(x_k + u_k - w_k) \right\} \right], \qquad k = 0, \ldots, N-1.$$

Define $y_k = x_k + u_k$ and $G_k(y) = E\{J_{k+1}(y - w)\}$. Then

$$\begin{aligned} J_k(x_k) &= r_k(x_k) + \min_{y_k \geq x_k} \left[E\{J_{k+1}(y_k - w_k)\} \right] \\ &= r_k(x_k) + \min_{y_k \geq x_k} G_k(y_k), \qquad k = 0, \ldots, N-1. \end{aligned}$$

Assume that J_{k+1} is convex. Then, since addition and taking expectation preserves convexity, G_k is convex. If we also assume that $\lim_{|y| \to \infty} G_k(y) = \infty$, then G_k has an unconstrained minimum, denoted by S_k and the optimal policy has the form

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k, & \text{if } x_k < S_k, \\ 0, & \text{if } x_k \geq S_k, \end{cases}$$

where for each k, the scalar S_k minimizes $G_k(y)$. To show that J_k is convex and that $\lim_{|y| \to \infty} G_k(y) = \infty$ for all k, note that since r_N is convex, J_N is convex. Since the derivative of $r_N(x)$ goes to ∞ as $x \to \infty$ and to $-\infty$ as $x \to -\infty$, $J_N(x)$ and $G_{N-1}(y)$ go to ∞ as $|x|$ and $|y|$, respectively, approach ∞. Then the optimal policy at time $N-1$ is given by

$$\mu_{N-1}^*(x_{N-1}) = \begin{cases} S_{N-1} - x_{N-1}, & \text{if } x_{N-1} < S_{N-1}, \\ 0, & \text{if } x_{N-1} \geq S_{N-1}. \end{cases}$$

Now assume that J_{k+1} is convex and that $\lim_{|x| \to \infty} J_{k+1}(x) = \infty$. Then $\lim_{|y| \to \infty} G_k(y) = \infty$ and

$$J_k(x_k) = \begin{cases} r_k(x_k) + E\{J_{k+1}(S_k - w_k)\}, & \text{if } x_k < S_k, \\ r_k(x_k) + E\{J_{k+1}(x_k - w_k)\}, & \text{if } x_k \geq S_k. \end{cases}$$

Thus J_k is convex and $\lim_{|x| \to \infty} J_k(x) = \infty$.

b) The system now evolves as

$$x_{k+1} = x_k + u_{k-1} - w_k, \qquad k = 0, 1, \ldots, N-1.$$

Adding u_k to both sides and making the change of variable $y_k = x_k + u_{k-1}$, we have

$$y_{k+1} = y_k + u_k - w_k, \qquad k = 0, 1, \ldots, N-1.$$

The cost to minimize is

$$\begin{aligned} E\{\sum_{k=0}^{N} r_k(x_k)\} &= E\{r_0(x_0) + \sum_{k=0}^{N-1} r_{k+1}(x_{k+1})\} \\ &= E\{r_0(y_0 - u_{-1}) + \sum_{k=0}^{N-1} r_{k+1}(y_k - w_k)\} \\ &= E\{r_0(y_0 - u_{-1}) + \sum_{k=0}^{N-1} \hat{r}_k(y_k)\}, \end{aligned}$$

where

$$\hat{r}_k(y_k) = E_{w_k}\{r_{k+1}(y_k - w_k)\}, \qquad k = 0, \ldots, N-1.$$

By defining $\hat{r}_N(y_N)$ to be the constant $r_0(y_0 - u_{-1})$ for all y_N, our cost has the form of the problem of part (a).

Exercise 4.29

a) As suggested by the hint, consider the case where only one out of two available questions can be answered.

E[reward using order $1 \to 2$] $= p_1 R_1$

E[reward using order $2 \to 1$] $= p_2 R_2$

An appropriate index for this problem would thus be $p_i R_i$ as opposed to $p_i R_i/(1 - p_i)$.

b) We now have a problem that has N available questions, but the maximum number of questions that can be answered is $N - 1$.

Let A_i, for $i = 1, 2, \ldots, N$, be the set of all orderings with question i ordered last (meaning we never reach question i since it is ordered Nth). The sets A_i, $i = 1, 2, \ldots, N$, form a partition of the set of all possible orderings.

Consider the problem of maximizing expected reward over the set of orders A_j. Because question j is always last, this problem is equivalent to finding the optimal ordering for $N - 1$ questions (all questions except question j) without a limit on the number of questions that can be answered. Thus, the optimal ordering over the set A_j, denoted L_j, uses the index $p_i R_i/(1 - p_i)$ for $i \neq j$.

To find the optimal ordering over the set of all possible orderings, we only need to compare N orderings: choose ordering L_j that satisfies

E[reward using order L_j] $= \max_{L_i, i=1,2,\ldots,N}$ E[reward using order L_i]

Exercise 4.33

(a) We formulate this as a DP problem involving the following two states:

S: The state where the singer is satisfied just following a performance.

$\overline{S}$: The state where the singer is not satisfied just following a performance (but may still be placated to sing on the following night with the offer of a gift).

The initial state is S.

The transition probabilities are:

Without the offer of a gift:

$$p_{SS} = p, \qquad p_{S\overline{S}} = 1 - p,$$

$$p_{\overline{S}S} = 0, \qquad p_{\overline{S}\overline{S}} = 1.$$

With the offer of a gift:

$$p_{SS} = p, \qquad p_{S\overline{S}} = 1 - p,$$

$$p_{\overline{S}S} = qp, \qquad p_{\overline{S}\overline{S}} = 1 - qp.$$

Notice we may assume that the director never offers a gift when the singer is satisfied, meaning at state S, the cost-per-stage is 0. At state $\overline{S}$, the cost-per-stage is C without a gift offer, and $G + (1 - q)C$ with a gift offer.

The DP algorithm is

$$J_k(S) = pJ_{k+1}(S) + (1 - p)J_{k+1}(\overline{S}),$$

$$J_k(\overline{S}) = \min\left[C + J_{k+1}(\overline{S}),\, G + (1 - q)C + qpJ_{k+1}(S) + (1 - qp)J_{k+1}(\overline{S})\right],$$

$$J_N(S) = J_N(\overline{S}) = 0.$$

An alternative DP algorithm uses the singer's state at the beginning of a performance instead. Let T represent that the singer is satisfied at the beginning of a performance and $\overline{T}$ that the singer is not satisfied at the beginning of a performance (and will therefore not perform that night). Noting that the director's decision occurs after she has declared herself satisfied or not at the end of the performance, we have:

$$\tilde{J}_k(T) = p\tilde{J}_{k+1}(T) + (1 - p)\min\{\tilde{J}_{k+1}(\overline{T}), G + q\tilde{J}_{k+1}(T) + (1 - q)\tilde{J}_{k+1}(\overline{T})\},$$

$$\tilde{J}_k(\overline{T}) = C + \min\{\tilde{J}_{k+1}(\overline{T}), G + q\tilde{J}_{k+1}(T) + (1 - q)\tilde{J}_{k+1}(\overline{T})\},$$

$$\tilde{J}_N(T) = 0, \qquad \tilde{J}_N(\overline{T}) = C.$$

(b) The optimal policy is to offer a gift in state $\overline{S}$ at time k if

$$C + J_{k+1}(\overline{S}) \geq G + (1 - q)C + pqJ_{k+1}(S) + (1 - qp)J_{k+1}(\overline{S}),$$

or equivalently if

$$\frac{G - qC}{qp} \leq J_{k+1}(\overline{S}) - J_{k+1}(S).$$

We will now show by induction that

$$J_k(\overline{S}) - J_k(S) \geq J_{k+1}(\overline{S}) - J_{k+1}(S), \qquad k = 0, 1, \ldots, N - 1,$$

so that the threshold for offering a gift is lower in the early nights. This means, that the optimal strategy is specified by some time index $\overline{k}$: it is optimal to send a gift to placate a dissatisfied singer at all nights before night $\overline{k}$, and not to send at night $\overline{k}$ and any subsequent night.

Indeed, let

$$\beta_k = J_k(\overline{S}) - J_k(S).$$

We have $\beta_N = 0$ and from the DP algorithm, we have

$$\beta_k = \min\bigl\{C + p\beta_{k+1},\, G + (1 - q)C + p(1 - q)\beta_{k+1}\bigr\},$$

so we have $\beta_{N-1} \geq \beta_N$. The relation $\beta_k \geq \beta_{k+1}$ follows from the above equation and a simple induction.

We now define $\overline{k}$ using the fact that for $k \geq \overline{k}$, the optimal policy does not send a gift. For $k \geq \overline{k}$ we have the recursion $\beta_k = C + p\beta_{k+1}$, starting with $\beta_N = 0$. Solving this in closed form, we have $\beta_k = \frac{1-p^{N-k}}{1-p}C$ for $\overline{k} \leq k \leq N-1$. Therefore, we define $\overline{k}$ as the smallest integer k satisfying

$$\frac{G - qC}{qp} > \beta_{k+1} = \frac{1 - p^{N-k-1}}{1-p}C$$

or equivalently

$$\frac{G}{qC} > \frac{1 - p^{N-k}}{1-p}.$$

Notice if $G \leq qC$, then $\overline{k} = N$, which means it is optimal to send a gift at all stages. If $G > qC$, then $\overline{k} \leq N-1$.
(c) Consider the case where q is not constant but rather is a function of the stage index, say q_k. We write the same DP algorithm, replacing q with q_k, and once again we let $\beta_k = J_k(\overline{S}) - J_k(S)$. The optimal policy is then to offer a gift in state $\overline{S}$ and stage k if $G \leq q_k(C + p\beta_{k+1})$.

If q_k, which represents the probability of success of a gift, is a decreasing function in k, then intuitively the form of the optimal policy found in part (b) should still be optimal. More specifically, if the optimal policy when q_k is constant is to stop sending gifts at some point in time, then having q_k decrease over time should not make sending gifts at later stages (after stages at which it was optimal to not send) profitable.

Following the same procedure as in part (b), we may find the optimal policy is to send a gift when the singer is unsatisfied if and only if k is less than $\overline{k}$, where $\overline{k}$ is the smallest integer k satisfying

$$G > q_k C \frac{1 - p^{N-k}}{1-p}.$$

Notice that $\overline{k}$ exists because the right-hand side of the above equation is decreasing in k. To verify that this policy is optimal, we may show by induction that $\beta_k = \frac{1-p^{N-k}}{1-p}C$ for $\overline{k} \leq k \leq N$ and $\beta_k \leq \frac{1-p^{N-k}}{1-p}C$ for $0 \leq k \leq \overline{k} - 1$.

Exercise 4.34

(a) This problem is the same as the "asset selling" problem on p.168 except it has the same state evolution as the "case of correlated prices" on p. 173. The resulting DP algorithm is then:

$$J_k(x_k) = \begin{cases} \max\{\underbrace{(1+r)^{N-k}x_k}_{sell}, \underbrace{E_{w_k}[J_{k+1}(\lambda x_k + w_k)]}_{do\ not\ sell}\} & \text{if } x_k \neq T \\ 0 & \text{if } x_k = T \end{cases}$$

$$J_N(x_N) = \begin{cases} x_N & \text{if } x_N \neq T \\ 0 & \text{if } x_N = T \end{cases}$$

(b) It turns out for each stage k, a threshold for x_k exists above which it is optimal to sell and below which it is optimal to not sell. These thresholds are decreasing as k increases.
Let $V_k(x_k) = \frac{J_k(x_k)}{(1+r)^{N-k}}$. The DP algorithm then becomes:

$$V_k(x_k) = \begin{cases} \max\{\underbrace{x_k}_{sell}, \underbrace{(1+r)^{-1}E_{w_k}[V_{k+1}(\lambda x_k + w_k)]}_{do\ not\ sell}\} & x_k \neq T \\ 0 & x_k = T \end{cases}$$

$$V_N(x_N) = \begin{cases} x_N & x_N \neq T \\ 0 & x_N = T \end{cases}$$

The optimal stopping set for stage $N-1$ is straightforward to find:

$$
\begin{aligned}
V_{N-1}(x) &= \max\{x, (1+r)^{-1} E_{w_{N-1}}[\lambda x + w_{N-1}]\} \\
&= \max\{x, \frac{\lambda x + \bar{w}}{1+r}\} \\
T_{N-1} &= \{x \mid x \geq \frac{\lambda x + \bar{w}}{1+r}\} \\
&= \{x \mid x \geq \alpha_{N-1}\}
\end{aligned}
$$

where $\alpha_{N-1} = \frac{\bar{w}}{1+r-\lambda}$ and $\bar{w} = E[w_k]$ for all k. Depending on the magnitude of w_k relative to λ, x_{k+1} can be either greater or less than x_k, therefore T_{N-1} is not absorbing. We show by induction that $V_k(x)$ is convex, has a slope of 1 as x approaches infinity, and has a positive slope less than 1 as x approaches negative infinity.

We first show a base case. $V_{N-1}(x)$ is the maximum of affine functions, which is convex. $V_{N-1}(x) = x$ for $x \geq \alpha_{N-1}$, meaning $V_{N-1}(x)$ has slope 1 as $x \to \infty$. $V_{N-1}(x) = \frac{\lambda x + \bar{w}}{1+r}$ for $x < \alpha_{N-1}$, meaning $V_{N-1}(x)$ has slope $\frac{\lambda}{1+r} < 1$ as $x \to -\infty$.

We now assume for induction that $V_{k+1}(x)$ is convex, has a slope of 1 as $x \to \infty$, and has a positive slope less than 1 as $x \to -\infty$. Consider

$$V_k(x) = \max\{x, h_k(x)\},$$

where $h_k(x) = (1+r)^{-1} E_{w_k}[V_{k+1}(\lambda x + w_k)]$. We note that $h_k(x)$ is convex because performing a linear transformation, taking the expectation, and multiplying by a positive constant all preserve convexity. Because $0 < \frac{\lambda}{1+r} < 1$, $h_k(x)$ has slope $\frac{\lambda}{1+r}$ as $x \to \infty$ and some positive slope less than 1 as $x \to -\infty$. Given the form of $h_k(x)$, we then know that $h_k(x)$ and the function x have exactly one intersection point, say α_k, and that

$$V_k(x) = \begin{cases} x & x > \alpha_k \\ h_k(x) & x < \alpha_k \end{cases}$$

This form of $V_k(x)$ corresponds to the optimal policy:

$$
\begin{aligned}
&\textit{sell if } x_k > \alpha_k \\
&\textit{do not sell if } x_k < \alpha_k
\end{aligned}
$$

Notice that $V_k(x)$ is convex, has a slope of 1 as $x \to \infty$, and has a positive slope less than 1 as $x \to -\infty$.

Noting that $V_{N-1}(x) = \max\{x, (1+r)^{-1} E_{w_{N-1}}[\lambda x + w_{N-1}]\} \geq x = V_N(x)$ for all x, we have by the monotonicity property that $V_k(x) \geq V_{k+1}(x)$ for all x, $k = 0, 1, \ldots, N-1$, meaning $h_{k-1}(x) \geq h_k(x)$ for all x, $k = 1, 2, \ldots, N-1$. Therefore, $\alpha_k \geq \alpha_{k+1}$ for $k = 0, 1, \ldots, N-1$ (which is also apparent because we know $T_k \subset T_{k+1}$ for all k).

(c) The optimal policy remains the same. Assume we can sell a fraction β of the stock on one day and the fraction $(1-\beta)$ on a different day. Then it is optimal to sell when the value of the stock exceeds $\beta\alpha_k$ and $(1-\beta)\alpha_k$ respectively. But this will happen simultaneously, so if it is optimal to sell the fraction β it is also optimal to sell the fraction $(1-\beta)$.