Resistive circuit analysis. Kirchhoff's Laws

Fundamentals of DC electric circuits.

A simple model that we can use as a starting point for discussing electronic circuits is given on **Figure 1**.

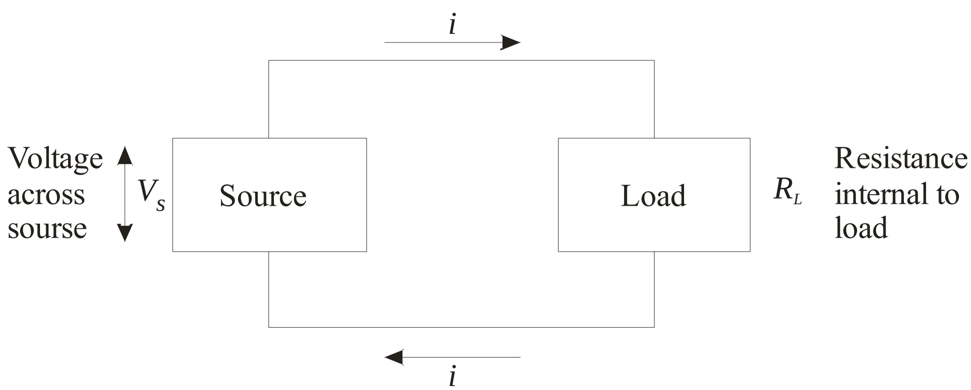


Figure 1. Fundamental circuit model

This circuit is made up of a source which provides a voltage across its terminals, labeled V_S and a load connected to the source which presents a resistance R_L to the current i flowing as indicated around a closed loop.

In order to characterize the operation of this circuit we must determine:

- What voltage V_S does the source provide as a function of current i?
- What resistance R_L does the load present?

In order to completely define the problem we have to establish the relationship between the voltage V_S, the resistance R_L and the current i. Before proceeding let's define the physical significance of these new physical variables and establish ways to represent them.

Current i

The current i results from the flow of electric charge around the closed loop shown on Figure 1. Electrons are electrically (negatively) charged particles and their flow in conductors such as wires results in electric current.

The current, i, is equal to the amount of charge, Q, passing through a cross-section per second and it is expressed as

$$i = \frac{dQ}{dt} \tag{1.1}$$

The unit of charge is the Coulomb. One Coulomb is equivalent to 6.24 x 10^{18} electrons. The unit for current is the ampere, A. One ampere = 1 Coulomb/sec.[1]

Voltage

In order to move electrons along a conductor some amount of work is required. The work required must be somehow supplied by an electromotive force usually provided by a battery or similar device. This electromotive force is referred to as the voltage or **potential difference** between two points or across an element. By representing an element with the block diagram shown on **Figure 2**, the voltage across the element represents the potential difference between terminals *a* and *b*. Mathematically the voltage v_{ab} is given by

$$v_{ab} = \frac{dW}{dQ} \tag{1.2}$$

where work (*W*) is measured in Joules and the charge (Q) in Coulombs.
The voltage is measured in volts (V) and 1 volt $\equiv 1 \ \frac{\text{Joule}}{\text{Coulomb}} = 1 \frac{\text{Newton meter}}{\text{Ampere second}}$.

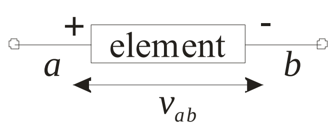


Figure 2. Voltage across an element

The positive (+) and negative (-) signs shown on **Figure 2** define the polarity of the voltage v_{ab}. With this definition, v_{ab} represents the voltage at point *a* relative to point *b*. Equivalently we may also say that the voltage at point *a* is v_{ab} volts higher than the voltage at point *b*.

[1] The SI system of units is based on the following seven base units:

Length:	m
Mass:	kg
Time:	s
Thermodynamic temperature:	K
Amount of substance:	mol
Luminous intensity:	cd
Current	A

The purpose for this small diversion is to remind us of the power of dimensional analysis in engineering.

i/v **curves**

The two dynamical variables of electronic circuits are current and voltage. It is useful therefore to explore the characteristic relationship between these for various circuit elements. The relationship between voltage and current for an element or for an entire circuit as we will explore shortly is fundamental in circuit design and electronics. We will start this exploration by looking at the i/v space of the two most fundamental sources: the voltage source and the current source.

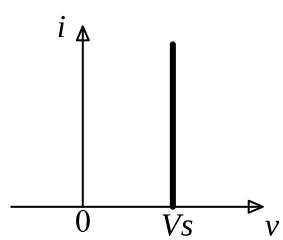
Ideal DC voltage sources

The most common voltage source is a battery. The voltage provided by a battery is constant in time and it is called DC voltage. In its ideal implementation the battery provides a specific voltage at all times and for all loads.

The common symbols for a battery are shown on **Figure 3**.

Figure 3. Battery symbols

The i/v curve of an ideal battery is:



As the i/v curve shows, regardless of the current flowing through the battery, the voltage across the battery remains constant. The actual amount of current that is provided by the battery depends on the circuit that is connected to the battery.

This is not a realistic model of a battery. Real batteries contain small internal resistors resulting in a modification of the i/v curve. We will look at these non-ideal effects in more detail shortly.

Ideal DC current sources

The current source is a device that can provide a certain amount of current to a circuit. The symbol for a DC current source and the i/v characteristic curve of an ideal current source are shown on Figure 4.

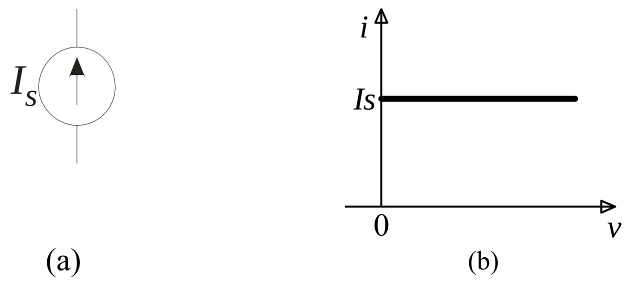


Figure 4. (a) Symbol of current source and (b) i/v characteristic curve of ideal current source.

Ideal resistor

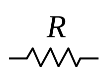
An ideal resistor is a passive, linear, two-terminal device whose resistance follows Ohm's law given by,

$$\boxed{v = iR} \tag{1.3}$$

which states that the voltage across an element is directly proportional to the current flowing through the element. The constant of proportionality is the resistance R provided by the element. The resistance is measured in Ohms, Ω, and

$$1\Omega = 1\frac{\text{V}}{\text{A}} \tag{1.4}$$

The symbol for a resistor is,



Notice that there is no specific polarity to a physical resistor, the two leads (terminals) are equivalent.

The circuit shown on Figure 5 consists of a voltage source and a resistor. These two elements are connected together with wires which are considered to be ideal. The current flowing through the resistor is given by

$$i = \frac{V_S}{R} \tag{1.5}$$

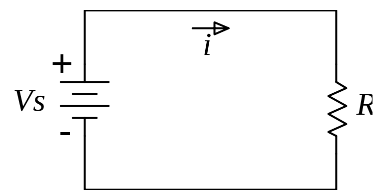


Figure 5. Simple resistive circuit.

The i/v curve for a resistor is a straight line (the current is directly proportional to the voltage). The slope of the straight line is $\frac{1}{R}$ (see Figure 6) For convenience we define the conductance (G) of a circuit element as the inverse of the resistance.

$$i = \frac{1}{R}v = Gv \tag{1.6}$$

The SI unit of conductance is the siemens (S)

$$S = \frac{1}{\Omega} = \frac{\text{A}}{\text{V}} \tag{1.7}$$

The most important use of i/v curves is to characterize a component or an entire circuit as we will see later. The i/v curve of the resistor shown on Figure 6 describes how that resistor will behave for any voltage or current. We can therefore use the i/v curve to find the operating points of circuits. For our circuit (Figure 5) the voltage is set by the battery at V_S and thus the operating point may be determined as shown graphically on Figure 6.

The power of this method should not be dismissed just because of its apparent simplicity. The i/v curve is one of the most powerful tools for circuit analysis and we will use it extensively in characterizing circuits and electronic components.

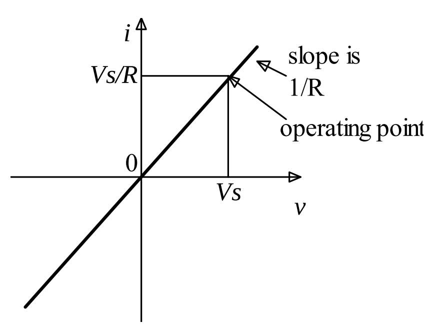


Figure 6. i/v curve of a resistor

The linear i/v relationship of the resistor does not hold for very high voltages and currents. The finite range of operation is in practice limited by the power that a resistor can dissipate.

Power

When an electric current flows through a resistor, energy is irreversibly lost (dissipated) in overcoming the resistance. This dissipated power shows up as heat. Power is the rate at which energy is delivered

$$P = \frac{dW}{dt} \tag{1.8}$$

The units for power are Joules/sec or Watts, W. (1 Joules/sec = 1 W)

Power can be related to voltage and current by rewriting Eq. (1.8) as,

$$P = \frac{dW}{dt} = \frac{dW}{dQ}\frac{dQ}{dt} = vi \tag{1.9}$$

By substituting Ohm's law in Eq. (1.9), the power dissipated in a resistor of resistance *R* is a non-linear function of either *i* or *v* and is given by

$$P = i^2 R \quad \text{or} \quad P = \frac{v^2}{R} \tag{1.10}$$

Power rating is a fundamental constraint of resistors and electronic devices in general. The power rating of a resistor corresponds to the maximum power that the device can dissipate without adversely affecting its operation. When the power rating is exceeded the resistor overheats and it is destroyed by burning up.

The power rating constraint is represented in the i/v space by the thick curve shown on **Figure 7**.

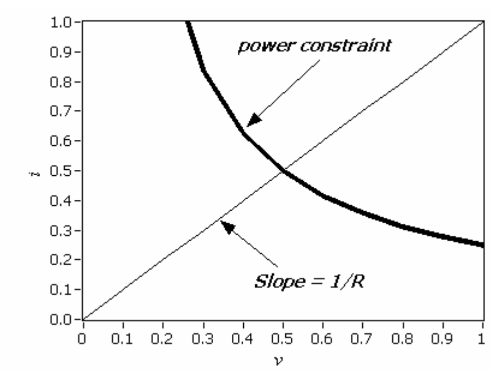


Figure 7. Resistor characteristic curve with power rating constraint

Circuits and Networks

By convention everything in a circuit is assumed to happen in the elements of a circuit, the lines just show the interconnections. Figure 8 represents a general circuit composed of elements *e1 … e5*. The elements could be any two terminal devices (voltage source, current source, resistor, capacitor, inductor, etc).

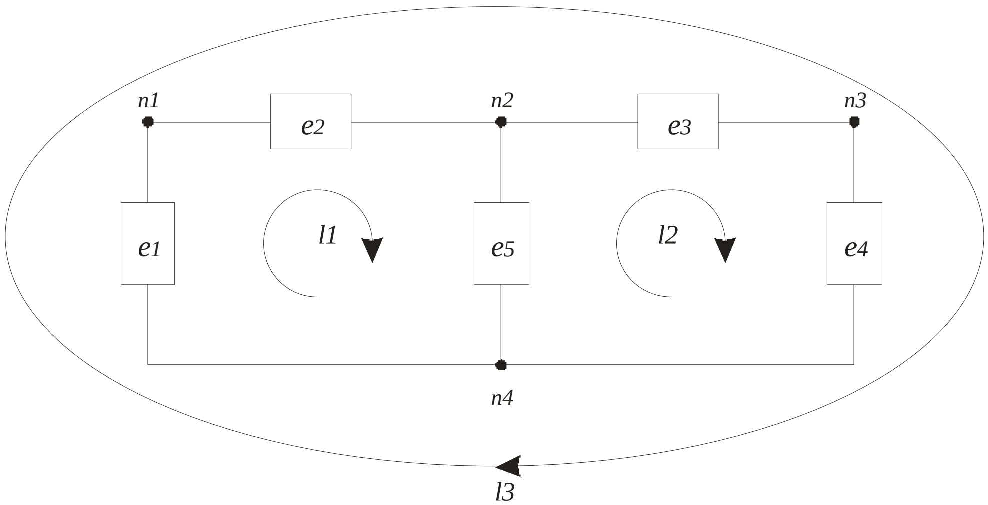


Figure 8. Example resistive network

The terminals of the various elements are connected together forming the nodes *n1 … n4* as indicated on Figure 8.

The connection between two elements is called a branch and the loops *l1, l2* and *l3* are closed connections of branches.

Kirchhoff's Laws

Kirchhoff's laws known as Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL) are based respectively on the conservation of charge and the conservation of energy and are derived from Maxwell's equations. They along with Ohm's law present the fundamental tools for circuit analysis.

Kirchhoff's Current Law states that: The current flowing out of any node in a circuit must be equal to the current flowing into the node. It is expressed mathematically as

$$\sum_{n=1}^{N} i_n = 0 \tag{1.11}$$

where N is the number of branches that are connected to the node. Consider the node shown on **Figure 9**. By adopting the sign convention that current flowing ito a node is positive (+) and current flowing out of the node is negative (-), application of KCL gives

$$i1 + i2 = i3 + i4 \tag{1.12}$$

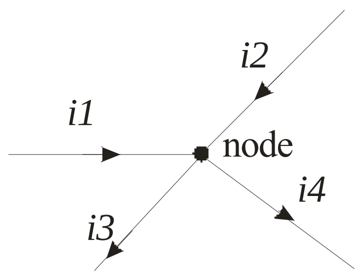


Figure 9. A node with multiple currents flowing into and out of it.

The definition of KCL may be extended to say that the algebraic sum of currents flowing into and out of a system must be zero. **Figure 10** illustrates a closed boundary and the currents entering and leaving the boundary.

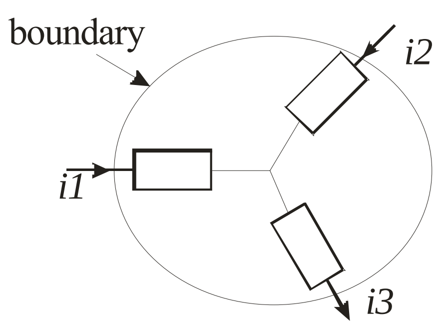


Figure 10. KCL applied to currents entering and leaving a closed boundary

Here again application of KCL gives

$$i1 + i2 = i3 \tag{1.13}$$

When a current source is present on a certain branch the current at that branch is equal to the current of the source. As an example consider the situation illustrated on **Figure 11**.

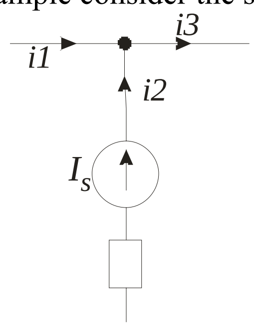


Figure 11. Current constraint by a current source.

Application of KCL at the indicated node gives: $i1 + i2 = i3$
But the current $i2$ is forced by the current source to be equal to I_S. Therefore, KCL reduces to $i1 + I_S = i3$ which indicates that the unknown current *i2* has been removed from the KCL equation by virtue of the current source constraint.

Kirchhoff's Voltage Law states that: The algebraic sum of voltages around a closed loop is zero. It is expressed mathematically as

$$\sum_{n=1}^{N} v_n = 0 \tag{1.14}$$

where N is the number of voltages in the loop. The number of voltages is equal to the number of elements encountered as we go around the loop.

Figure **12** shows a single loop circuit that we are going to use to illustrate KVL. By going around the loop in the indicated clockwise direction the sum of the voltages encountered is

$$v1 + v2 + V2 + V1 = 0 \tag{1.15}$$

Which is equivalent to

$$v1 + v2 = -V1 - V2 \tag{1.16}$$

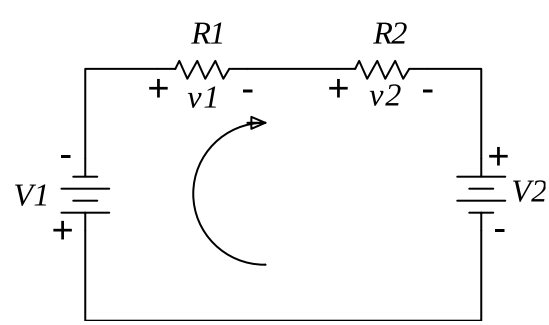


Figure 12. Single loop resistive circuit.

Now let us solve for the circuit shown on Figure 13 by applying KCL and KVL.

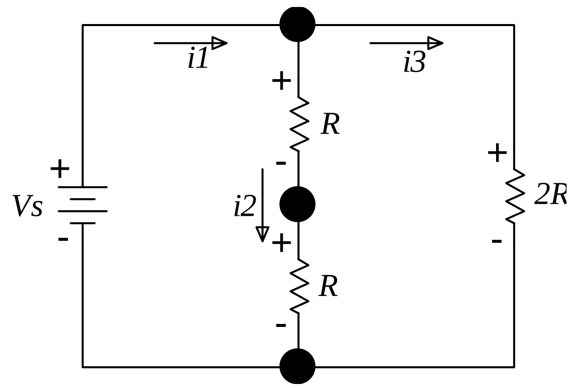


Figure 13. Example resistive circuit.

Apply Kirchhoff's laws.

- Let's start with the nodes.

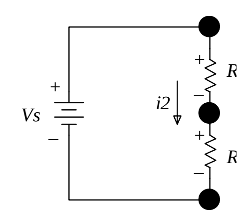
  1. The middle node is already accounted for since we assigned the current above and below it the same value, *i2*. This is just Kirchhoff's current law which says that the current going into a node is equal to that going out.

  2. The bottom and top nodes are exactly the same and KCL for them is,

$$\begin{aligned} & i1 - i2 - i3 = 0 \\ & or \\ & i1 = i2 + i3 \end{aligned} \tag{1.17}$$

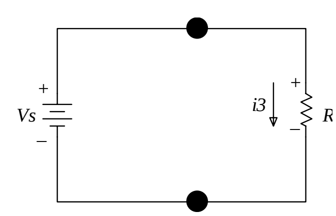
- Next let's apply KVL to the three loops

  1. Loop 1: voltage source *V1* and 2 resistors,



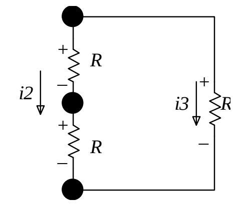
$$\begin{aligned} & i2R + i2R - Vs = 0 \\ & or \\ & i2 = \frac{Vs}{2R} \end{aligned} \tag{1.18}$$

  2. Loop 2: voltage source *V2* and 1 resistor,



$$\begin{aligned} & 2R\,i3 - Vs = 0 \\ & or \\ & i3 = \frac{Vs}{2R} \end{aligned} \tag{1.19}$$

3. Loop 3: 3 resistors,



$$2Ri3 - i2R - i2R = 0$$
$$or \qquad (1.20)$$
$$\left(i3 - i2\right)R = 0$$

Recall that the unkowns in the problem are: *i1, i2* and *i3* and the equations that we just derived are:

$$i1 = i2 + i3 \qquad (1.21)$$

$$i2 = \frac{Vs}{2R} \qquad (1.22)$$

$$i3 = \frac{Vs}{2R} \qquad (1.23)$$

$$(i2 - i3)R = 0 \qquad (1.24)$$

As we see one Eq. (1.24) is redundant. Indeed in the above example where we applied Kirchhoff's laws to every node and loop we ended up with more equations than we need to solve for the voltages and currents. For now this simply means that there are many ways to solve the same problem and all of them will lead to the same result. For simple circuits this is not a problem since you can see your way through the entire analysis. For complex circuits it becomes important to have a more formal means of applying Kirchhoff's laws so that one complete set of simultaneous equations are generated. We will discuss this in the next lecture.

Voltage divider: Series Connection of Resistors

The voltage divider circuit is the most convenient means of passively stepping down the voltage from a fixed voltage source. The circuit is shown on **Figure 14**.

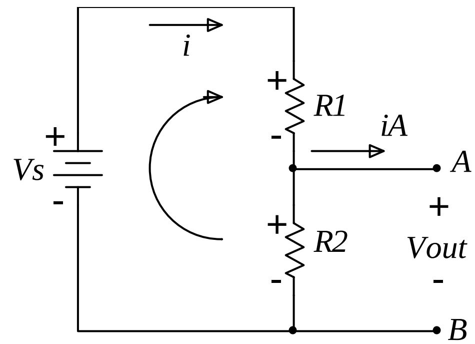


Figure 14. Voltage divider circuit

We would like to determine the voltage Vout measured across the terminals A and B.

We will calculate *Vout* by assuming that nothing is connected across terminals A and B. So, the output is *open-circuited* and thus no current flows ($iA = 0$). Voltage *Vout* is thus the voltage drop across resistor *R2.* Therefore by finding the current flowing through R2 we determine *Vout* by applying Ohm's law.

$$V_{out} = iR2 \tag{1.25}$$

We can find the current *i* by applying KVL around the loop as indicated on **Figure 14**.

$$\begin{aligned} &iR1 + iR2 = Vs \\ &or \\ &i = \frac{Vs}{(R1+R2)} \end{aligned} \tag{1.26}$$

And the output voltage Vout becomes:

$$V_{out} = v_{R2} = i\,R2 = Vs\left(\frac{R2}{R1+R2}\right) \tag{1.27}$$

Similarly, the voltage across resistor R1 is

$$v_{R1} = i\,R1 = Vs\left(\frac{R1}{R1+R2}\right) \tag{1.28}$$

And as expected is we add Eq. (1.27) and Eq. (1.28) we obtain

$$v_{R2} + v_{R1} = Vs \tag{1.29}$$

By considering a circuit where we are using N resistors connected in series as indicated on **Figure 15** we can show that the total resistance seen by the current *i* is the sum of the resistances.
Applying KVL around the loop shown on **Figure 15** gives

$$\begin{aligned} Vs &= iR1 + iR2 + \ldots + iRN \\ &= i(R1 + R2 + \ldots + RN) \\ &= i\,\text{Req} \end{aligned} \tag{1.30}$$

where Req, the equivalent resistance is given by

$$\text{Req} = \sum_{n=1}^{N} \text{R}n = R1 + R2 + \ldots + RN \tag{1.31}$$

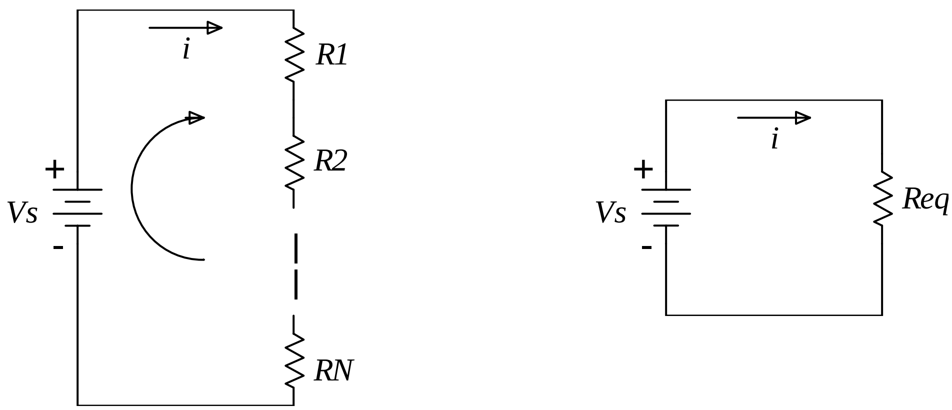


Figure 15. Series connection of resistors

Current Division: Parallel connection of resistors

The schematic on Figure 14 shows a simple current divider circuit. Here the two resistors *R1* and *R2* are connected in parallel. Lets determine the current *i1* and *i2* flowing in the two resistors.

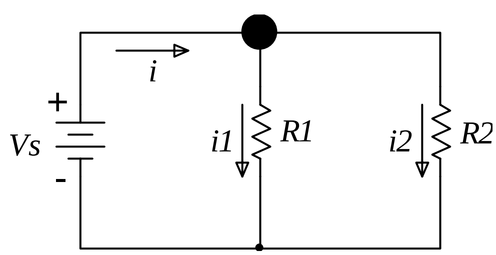


Figure 14. Current divider circuit.

There is one interesting node at which a current *i* flows in and $i_1 + i_2$ flows out. KCL then gives,

$$i = i1 + i2 \tag{1.32}$$

There three loops of interest as shown on **Figure 16**.

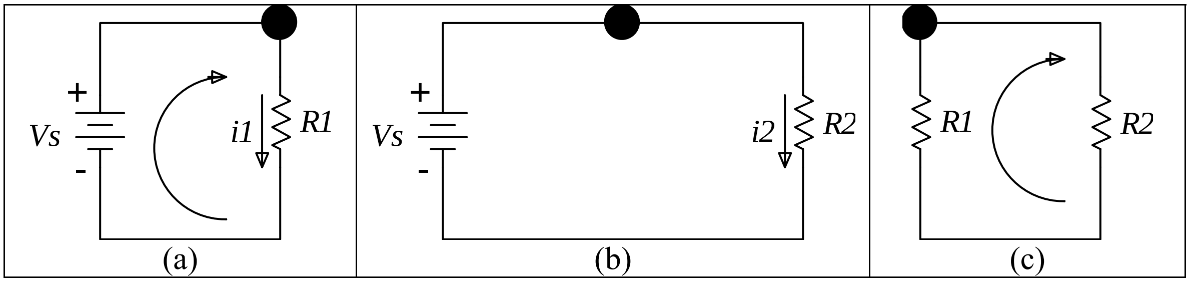


Figure 16. Loops of current divider circuit.

From **Figure 16** loops (a) and (b) that contain the voltage source and one of the resistors are of interest. We can write down KVL for each of these loops as:

$$V_s = i1R1 \tag{1.33}$$

And

$$V_s = i2R2 \tag{1.34}$$

Notice that the voltage across each resistor is the same. By dividing Eq. (1.33) and Eq. (1.34) we see that the ratio of the currents does not depend on the voltage.

$$\frac{i1}{i2} = \frac{R2}{R1} \tag{1.35}$$

The total current is given by

$$i = i1 + i2 = V_s\left(\frac{1}{R1} + \frac{1}{R2}\right) = V_s\frac{R1+R2}{R1R2} \tag{1.36}$$

And the currents i1 and i2 are

$$i1 = i\frac{R2}{R1+R2} \tag{1.37}$$

$$i2 = i\frac{R1}{R1+R2} \tag{1.38}$$

Therefore, the current through each leg (path) of the current divider is inversely proportional to the resistance over that leg.

The total current given by Eq. (1.36) provides a simple means of calculating the equivalent resistance for two resistors in parallel. From Ohm's law we identify the equivalent resistance as:

$$\frac{1}{\mathrm{R}_{eq}} = \frac{1}{R1} + \frac{1}{R2} = \frac{R1 + R2}{R1R2}$$
$$or \tag{1.39}$$
$$R_{eq} = \frac{R1R2}{R1 + R2}$$

Which shows that resistors in parallel combine as rates.

For N resistors connected in parallel the equivalent resistance is

$$\frac{1}{\mathrm{R}_{eq}} = \frac{1}{R1} + \frac{1}{R2} + \ldots \frac{1}{RN} \tag{1.40}$$

Note that the equivalent resistance R_{eq} is smaller than the smallest resistance in the parallel arrangement.

The equivalent conductance of N resistors connected in parallel is

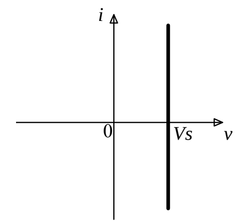
$$G_{eq} = G1 + G2 + \ldots GN \tag{1.41}$$

where $Gn = \frac{1}{Rn}$. The current flowing through resistor R_n is

$$i_n = i\frac{Gn}{G_{eq}} \tag{1.42}$$

Non-ideal voltage sources

An ideal voltage source delivers any current and it has the *i* / *v* curve shown on the figure to the right.



This implies that the voltage source could deliver any amount of power ($P = iv$), an unphysical condition.

A real voltage source has an internal resistor that limits the amount of current that can be delivered. A model of a non-ideal voltage source with an internal resistance *Rs* connected to a load resistance *Rl* is shown on **Figure 17**. The voltage source is enclosed by the dotted rectangle.

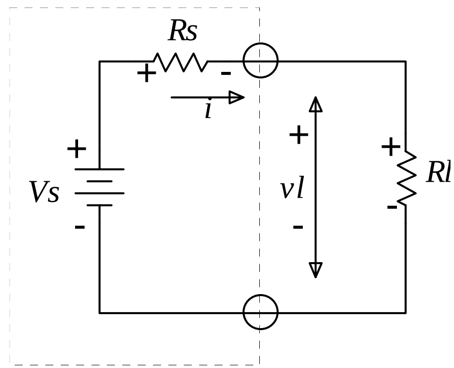


Figure 17. Non-ideal voltage source circuit

The operating characteristics of a real voltage source depends on both its internal (source) resistance, and the resistance of the network that it is connected to (the load).

We will plot out the *i/v* characteristic of the real voltage source as a function of the load resistance. So, we need to determine the voltage over the load resistor, *vl*, and the current through the load resistor as a function of the load resistance.

The source and load resistors are in series, and so the current can be calculated via Ohm's law with the equivalent resistance (*Rs* + *Rl*),

$$i = \frac{Vs}{(Rs + Rl)} \tag{1.43}$$

Given the current we can then calculate the voltage dropped across the load resistor.

$$vl = Vs\frac{Rl}{(Rs + Rl)} \tag{1.44}$$

The voltage *vl* is also the voltage appearing at the output terminals of the voltage source.

Notice that when *Rl* >> *Rs* then the real voltage source is very close to an ideal source. The presence of the source resistance limits the power that the real voltage source can provide. If we short circuit the voltage source the current output is

$$i_{\max} = \frac{Vs}{Rs} \qquad (1.45)$$

So the maximum power is

$$P_{\max} = Vs\, i_{\max} = \frac{Vs^2}{Rs} \qquad (1.46)$$

The smaller the source resistance the more ideal is a voltage source and the greater the power that it can deliver.

The *i/v* curve for a real voltage source is shown on **Figure 18**(a). The operating point is determined by the intersection of the load resistor line with the line defining the operating space of the voltage source **Figure 18**(b).

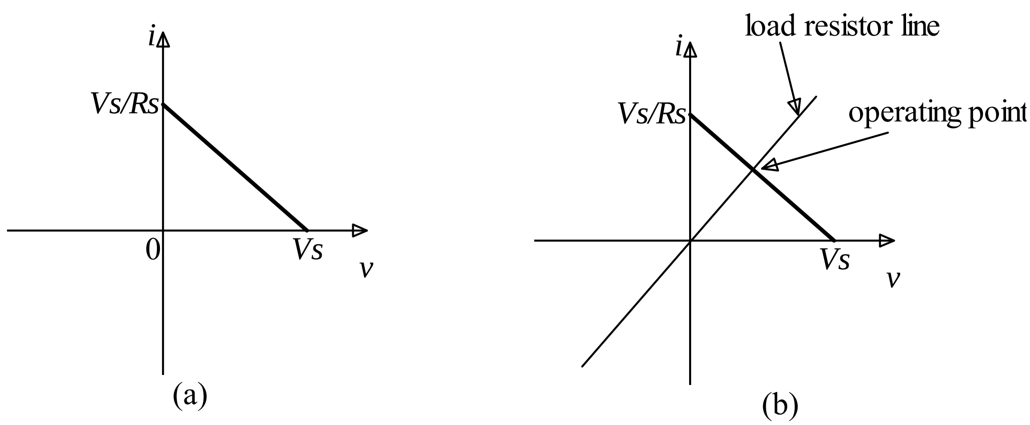


Figure 18. *i/v* characteristic curve of non-ideal voltage source.

It is useful to plot the ratio of vl/Vs versus Rl/Rs. **Figure 19** shows the plot which indicates that the actual voltage across *Rl* varies with the power demand of the load. When the load demands too much power from the source, then the load pulls down the voltage.

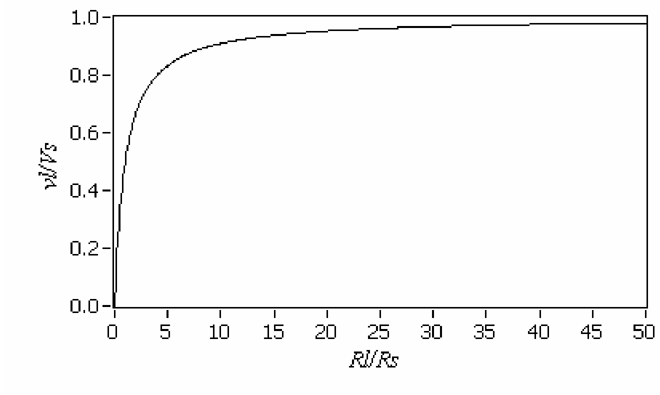
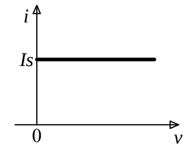


Figure 19. Performance characteristics of a voltage source

Note the general design principle; *a good voltage source has a source resistance that is much smaller than any load resistance it will see.*

Non-ideal current source

The ideal current source delivers a constant current into any load, and thus can assume any voltage difference across its terminals. The *i/v* curve of an ideal current source is shown on the figure to the right.



A real current source model includes a parallel resistor to define the voltage of the device and it is shown on **Figure 20** connected to a load resistor *R*l.

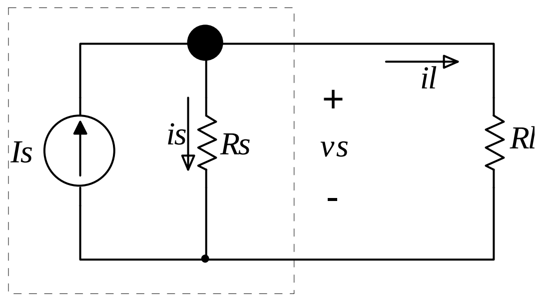


Figure 20. Non-ideal current source circuit

The circuit is a current divider, and applying KCL at the indicated node shows that the source current, *Is*, is divided into an internal current, *is*, and the load current, *il*.

In the open circuit configuration ($Rl \to \infty$) the current $il = 0$ and the voltage across the terminals is,

$$vs_{\max} = IsRs \tag{1.47}$$

In the short circuit configuration ($Rl = 0$), then the output current equals the source current and no current flows through *Rs*. Using these two results we can draw the *i/v* curve for the real current source shown on **Figure 21**(a) and (b). From **Figure 21**(b) we see that as the load resistance increases the operating point moves toward the point *IsRs* which is reached when $Rl \to \infty$.

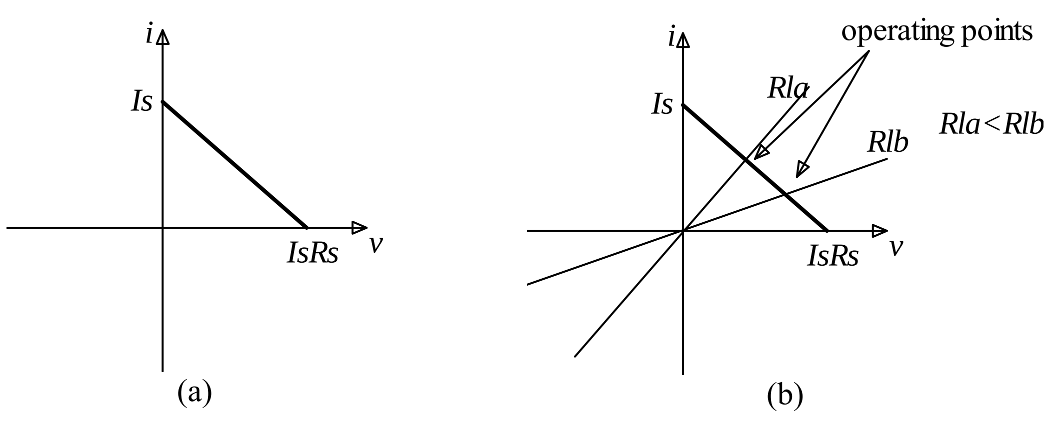


Figure 21. *i/v* characteristic curve of non-ideal current source.

To analyze the general case, apply KVL to the loop containing the two resistors of the circuit on **Figure 20**.

$$isRs = ilRl \tag{1.48}$$

and apply KCL to the node,

$$Is = is + il \tag{1.49}$$

then,

$$Is = il\frac{Rl}{Rs} + il$$

$$or \tag{1.50}$$

$$il = Is\left(\frac{Rs}{Rs + Rl}\right)$$

Figure 22 plots the ratio of the currents versus the ratio of the resistors,

$$\frac{il}{Is} = \frac{\frac{Rs}{Rl}}{\frac{Rs}{Rl} + 1} \tag{1.51}$$

And

$$\frac{v}{vs_{\max}} = \frac{1}{\frac{Rs}{Rl} + 1} \tag{1.52}$$

The current source approaches the ideal case when *Rs>>Rl*.

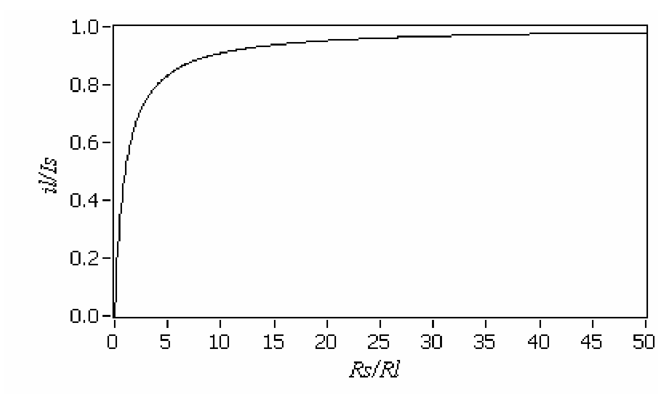


Figure 22. Plot il / Is as a function of the resistance ratio Rs / Rl

Notice that the good design rule for current sources is that the load resistor should be small compared to the source resistor.

Non-ideal resistors

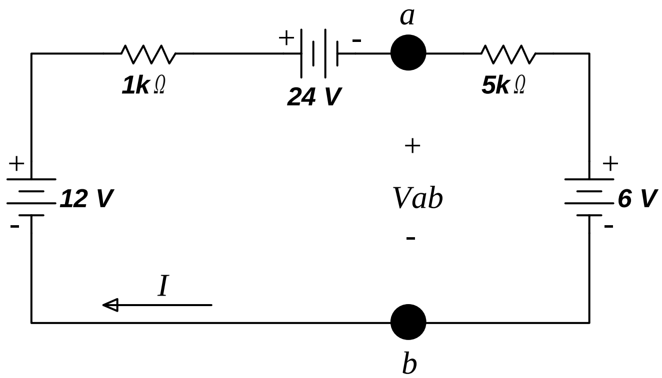
For an ideal resistor the i/v curve is a straight line regardless of the power dissipated in the device. Recall that $P=iR$, and that the power is dissipated in the form of heat. There are two important non-linearities with real resistors. First there is a finite temperature coefficient, so as real resistors heat up their resistance increases. The mathematical representation of this temperature dependence is

$$R = R_0\left[1+\alpha\Delta T\right] \tag{1.53}$$

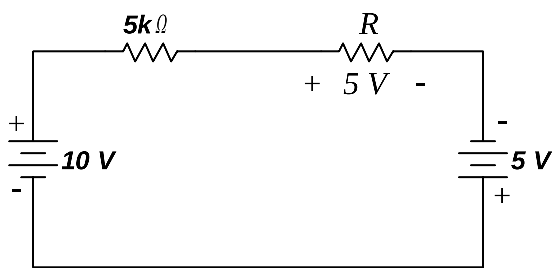
The parameter α is called the temperature coefficient and it has the units of degree^{-1}.

Practice problems:

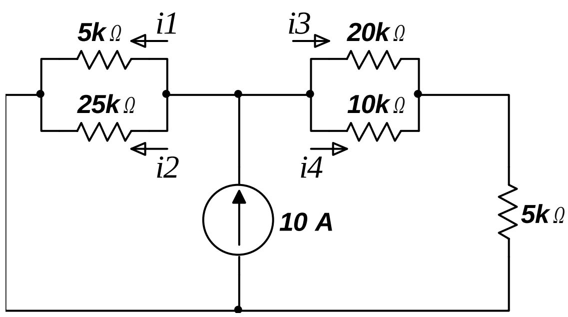
Calculate *Vab* and *I* for the following circuit.



Find R for the following circuit



Calculate the currents *i1*, …,*i4* in the following circuit.



Determine the currents i2 and i3 as a function of the other circuit parameters.