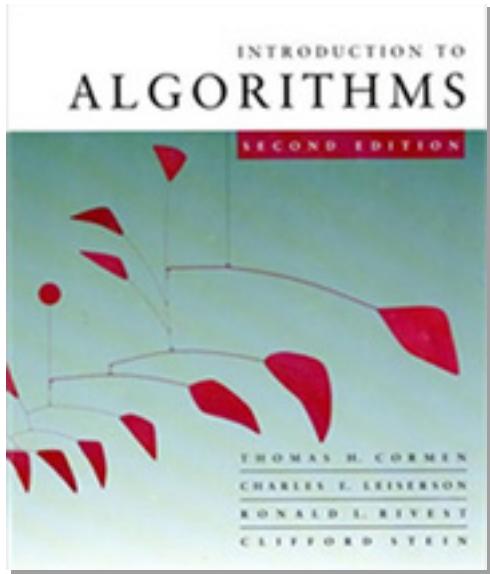


Introduction to Algorithms

6.046J/18.401J

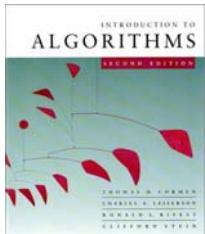


LECTURE 15

Dynamic Programming

- Longest common subsequence
- Optimal substructure
- Overlapping subproblems

Prof. Charles E. Leiserson

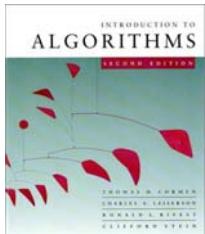


Dynamic programming

Design technique, like divide-and-conquer.

Example: **Longest Common Subsequence (LCS)**

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.

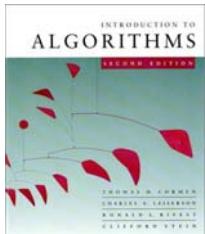


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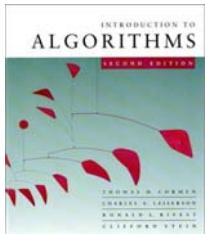
Example: **Longest Common Subsequence (LCS)**

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.

- “a” not “the”

x : A B C B D A B

y : B D C A B A



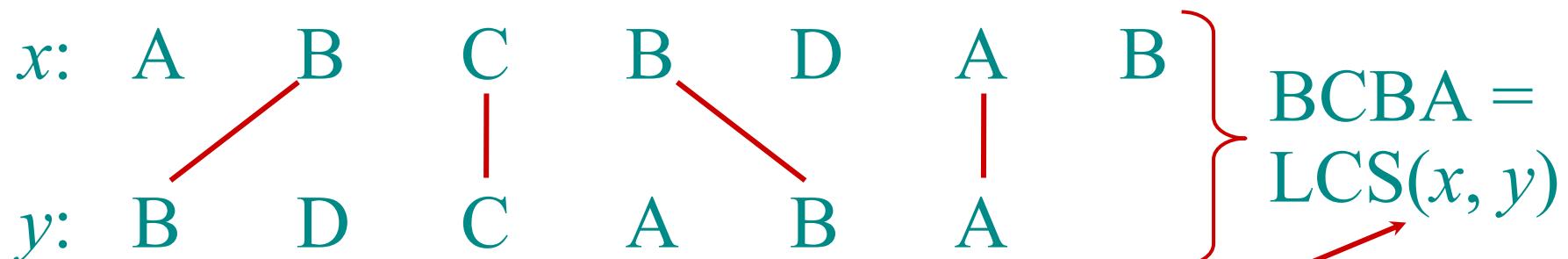
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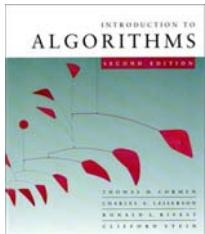
Example: *Longest Common Subsequence (LCS)*

- Given two sequences $x[1 \dots m]$ and $y[1 \dots n]$, find a longest subsequence common to them both.

“a” not “the”

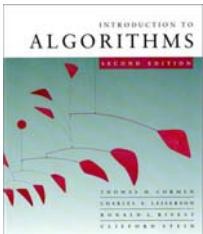


functional notation,
but not a function



Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.



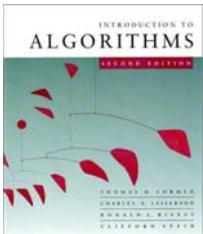
Brute-force LCS algorithm

Check every subsequence of $x[1 \dots m]$ to see if it is also a subsequence of $y[1 \dots n]$.

Analysis

- Checking = $O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

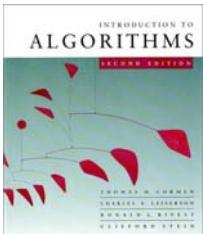
Worst-case running time = $O(n2^m)$
= exponential time.



Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

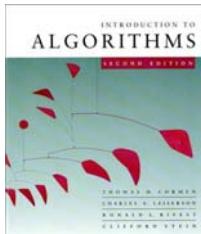


Towards a better algorithm

Simplification:

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Notation: Denote the length of a sequence s by $|s|$.



Towards a better algorithm

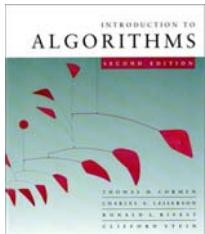
Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider *prefixes* of x and y .

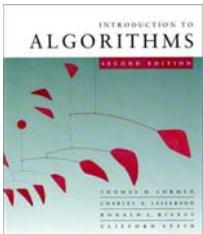
- Define $c[i, j] = |\text{LCS}(x[1 \dots i], y[1 \dots j])|$.
- Then, $c[m, n] = |\text{LCS}(x, y)|$.



Recursive formulation

Theorem.

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max \{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

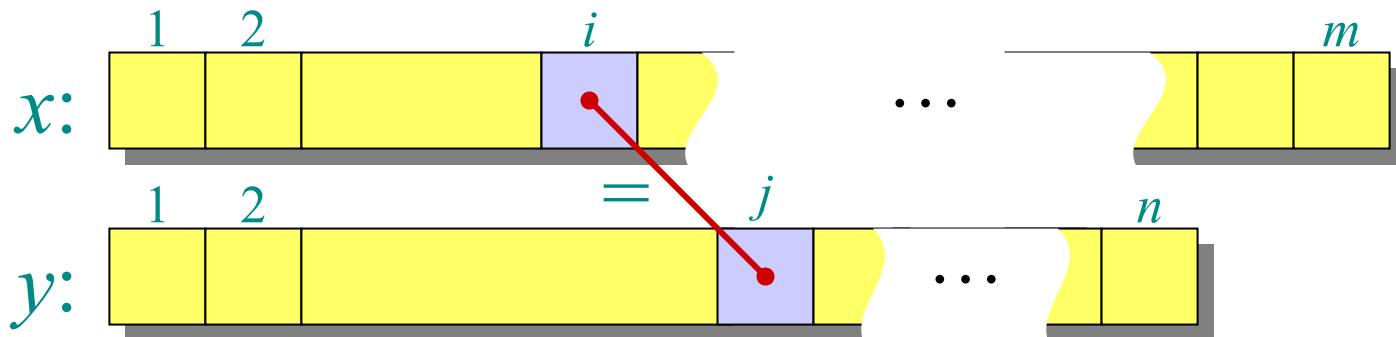


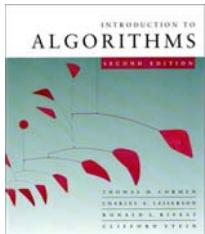
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Proof. Case $x[i] = y[j]$:



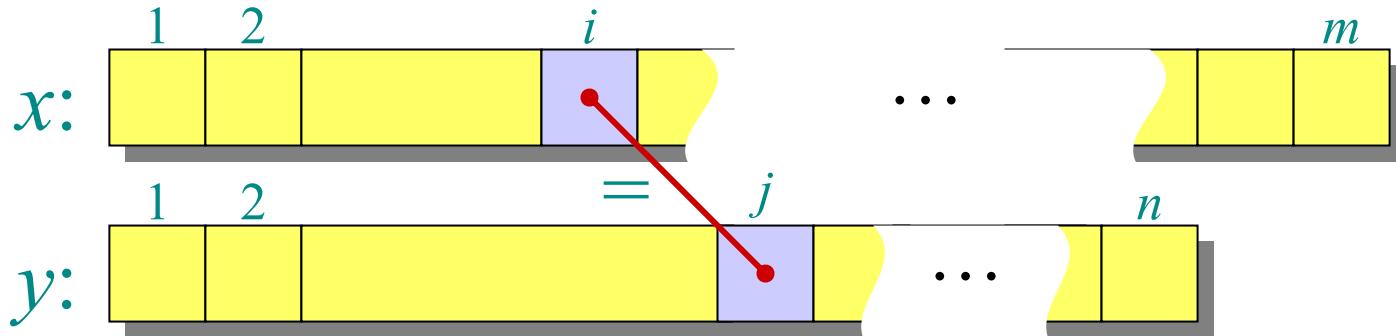


Recursive formulation

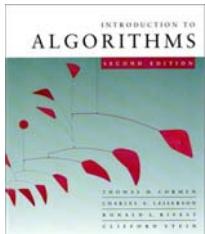
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Proof. Case $x[i] = y[j]$:



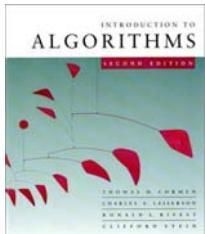
Let $z[1 \dots k] = \text{LCS}(x[1 \dots i], y[1 \dots j])$, where $c[i, j] = k$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \dots k-1]$ is CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$.



Proof (continued)

Claim: $z[1 \dots k-1] = \text{LCS}(x[1 \dots i-1], y[1 \dots j-1])$.

Suppose w is a longer CS of $x[1 \dots i-1]$ and $y[1 \dots j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \parallel z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1 \dots i]$ and $y[1 \dots j]$ with $|w \parallel z[k]| > k$. Contradiction, proving the claim.



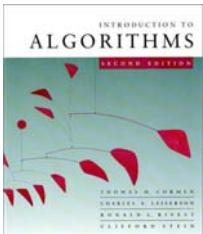
Proof (continued)

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Thus, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

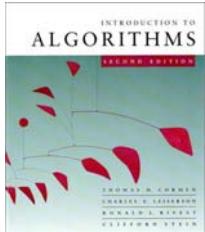
Other cases are similar.



Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

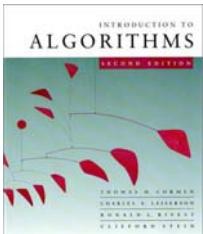


Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y .



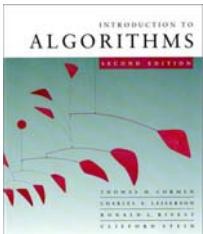
Recursive algorithm for LCS

$\text{LCS}(x, y, i, j)$

if $x[i] = y[j]$

then $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$

else $c[i, j] \leftarrow \max \{ \text{LCS}(x, y, i-1, j),$
 $\text{LCS}(x, y, i, j-1) \}$



Recursive algorithm for LCS

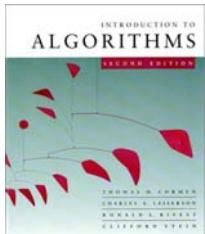
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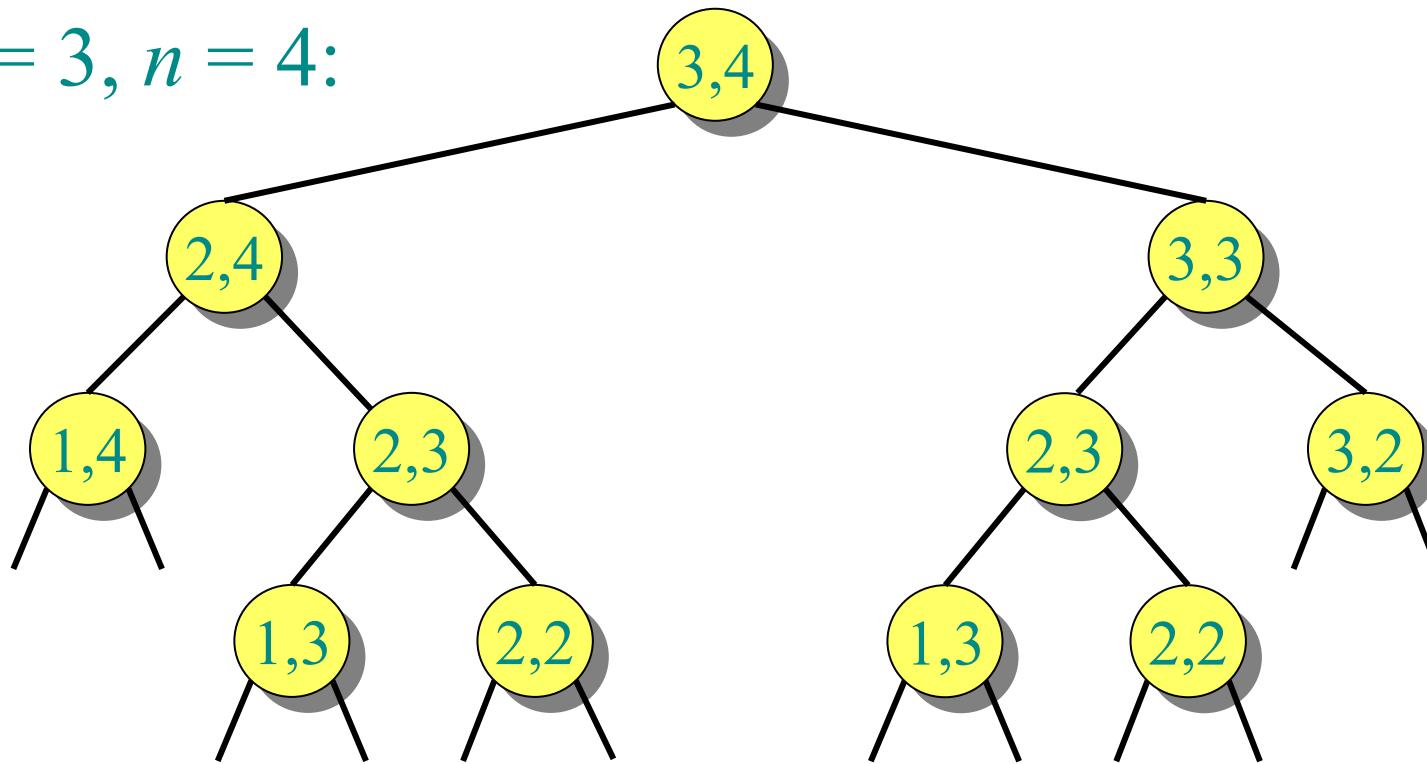
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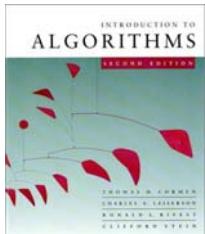
Worst-case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.



Recursion tree

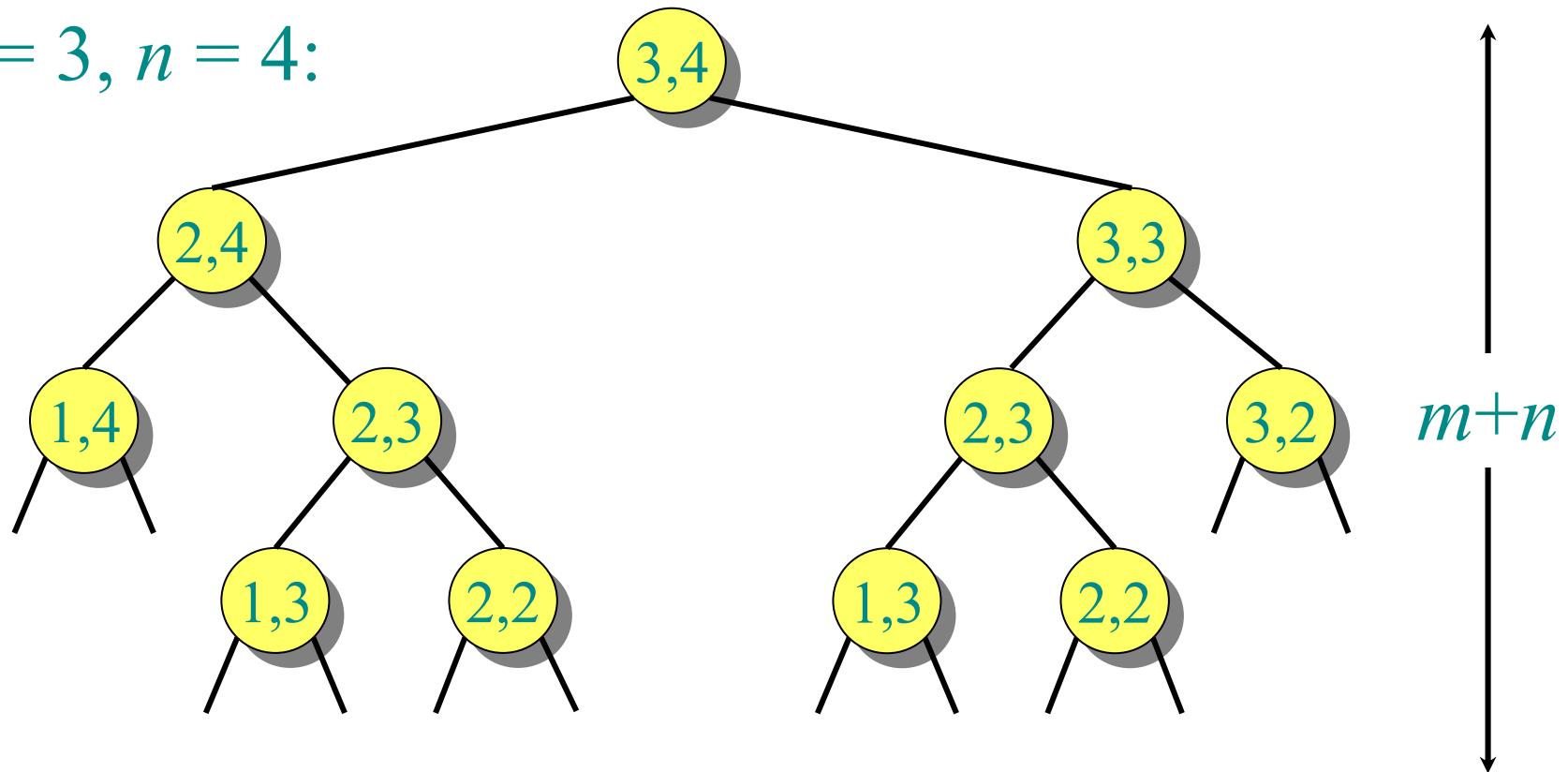
$m = 3, n = 4:$



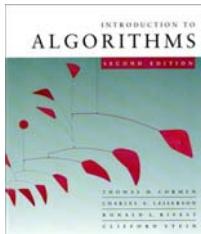


Recursion tree

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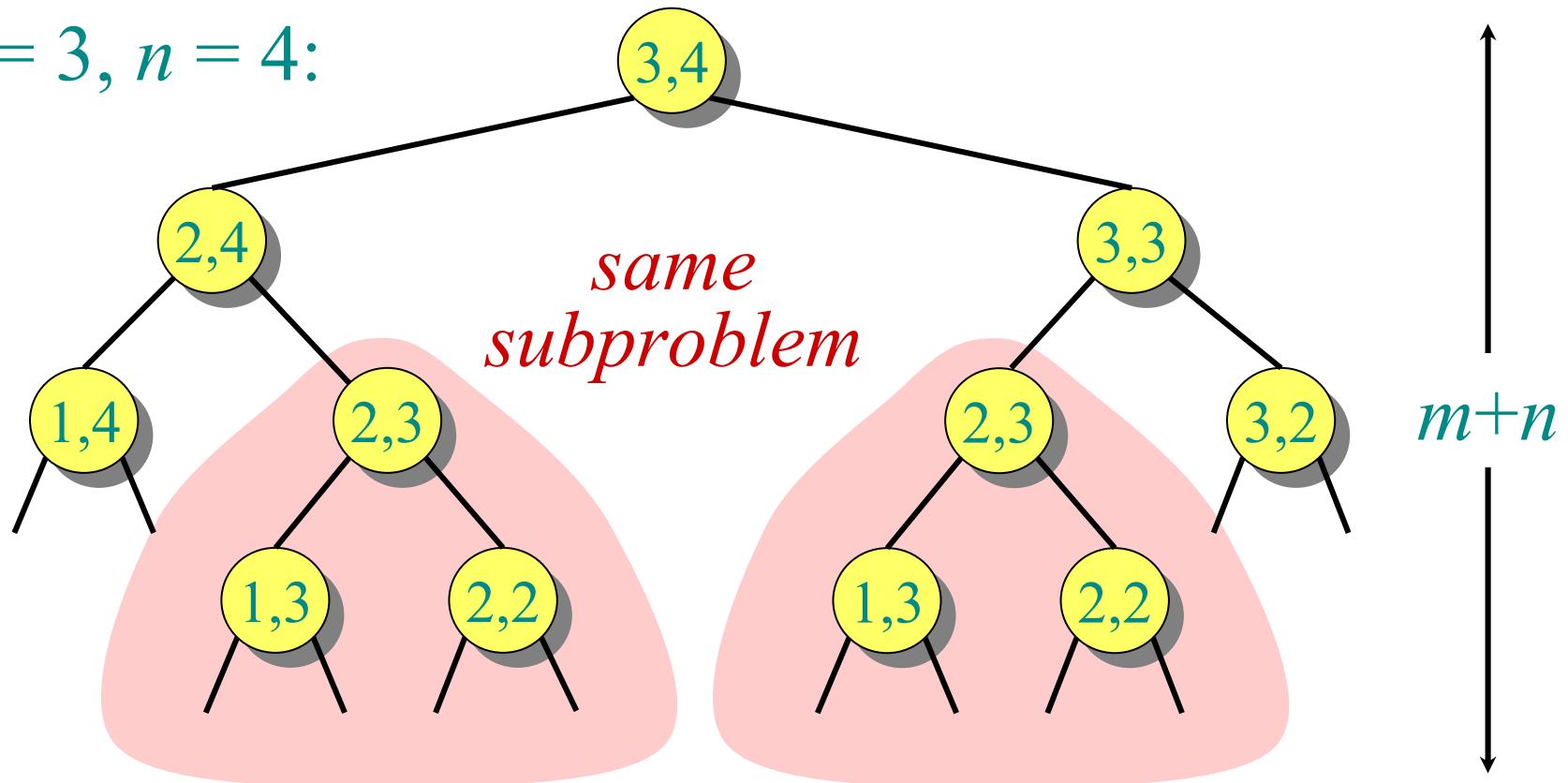


Height $= m + n \Rightarrow$ work potentially exponential.

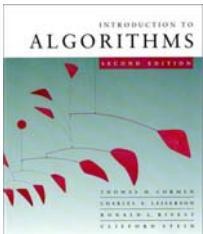


Recursion tree

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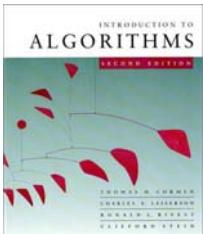
Height $= m + n \Rightarrow$ work potentially exponential,
but we're solving subproblems already solved!



Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a “small” number of distinct subproblems repeated many times.

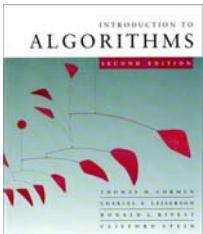


Dynamic-programming hallmark #2

Overlapping subproblems

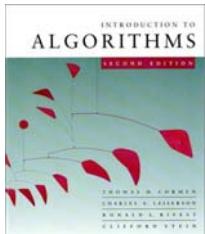
A recursive solution contains a “small” number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn .



Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.



Memoization algorithm

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$\text{LCS}(x, y, i, j)$

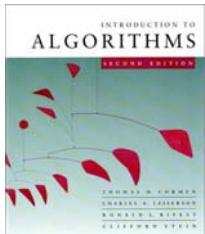
if $c[i, j] = \text{NIL}$

then if $x[i] = y[j]$

then $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$

else $c[i, j] \leftarrow \max \{ \text{LCS}(x, y, i-1, j), \text{LCS}(x, y, i, j-1) \}$

*same
as
before*



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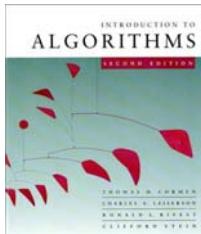
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*same
as
before*

Time = $\Theta(mn)$ = constant work per table entry.

Space = $\Theta(mn)$.



Dynamic-programming algorithm

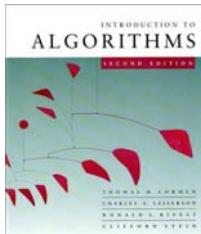
IDEA:

Compute the table bottom-up.

| | A | B | C | B | D | A | B |
|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| D | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 3 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |

The table illustrates a dynamic programming solution for a problem involving sequences A, B, C, and D. The rows and columns are labeled with these letters. The values in the cells represent the cost or weight of a path from the start to that cell. Diagonal arrows indicate the direction of computation, starting from the top-left and moving towards the bottom-right. The values are as follows:

- Row 0 (empty sequence): 0, 0, 0, 0, 0, 0, 0, 0
- Row A: 0, 1, 1, 2, 2, 2, 2, 3
- Row B: 0, 1, 2, 2, 3, 3, 3, 3
- Row C: 0, 0, 1, 2, 2, 2, 2, 2
- Row D: 0, 0, 1, 1, 1, 2, 2, 2
- Row B: 0, 0, 1, 1, 1, 1, 1, 1
- Row A: 0, 0, 0, 0, 0, 0, 0, 0



Dynamic-programming algorithm

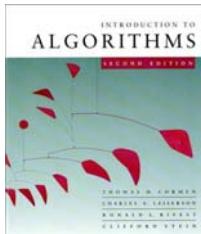
IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

| | A | B | C | B | D | A | B |
|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| D | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 3 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |

The table illustrates a dynamic programming solution for a problem involving strings A and B. The rows represent string A and the columns represent string B. The entries in the table are the lengths of the longest common subsequences (LCS) for substrings of A[i..j] and B[i..k]. The values are shown in green. Black diagonal slashes indicate that certain subproblems have been solved and are not being updated. The final value in the bottom-right corner is 4, representing the length of the LCS of the full strings A and B.



Dynamic-programming algorithm

IDEA:

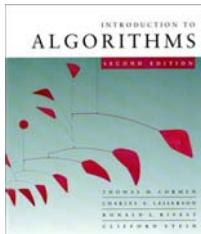
Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | A | B | C | B | D | A | B |
| B | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 3 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |

The table illustrates the computation of the Longest Common Subsequence (LCS) between two sequences, A and B. The rows represent sequence A and the columns represent sequence B. The diagonal elements (0,0), (1,1), (2,2), (3,3), and (4,4) are marked with a light purple color. Other cells contain numerical values representing the length of the LCS up to that point. Diagonal arrows indicate the path of the LCS from the bottom-left to the top-right. The final value in the bottom-right cell is 4, indicating the total length of the LCS.



Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

Space = $\Theta(mn)$.

Exercise:

$O(\min\{m, n\})$.

| | A | B | C | B | D | A | B |
|---|---|---|---|---|---|---|---|
| B | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| C | 0 | 0 | 1 | 2 | 2 | 2 | 2 |
| A | 0 | 1 | 1 | 2 | 2 | 2 | 3 |
| B | 0 | 1 | 2 | 2 | 3 | 3 | 3 |
| A | 0 | 1 | 2 | 2 | 3 | 3 | 4 |