Linear Programming

Examples: politics, flow, shortest paths

General form and converting to it

Simplex algorithm
- Iterating over slack form

Politics

How to campaign to win an election?

Staff estimates votes obtained per dollar spent advertising in support of a particular issue

	Policy	Demographic: Urban	Suburban	Rural
x_1	Building roads	-2	5	3
x_2	Gun control	8	2	-5
x_3	Farm subsidies	0	0	10
x_4	Gasoline tax	10	0	2

want to win majority in EACH demographic

	Urban	Suburban	Rural
Population	100,000	200,000	50,000
majority	50,000	100,000	25,000

by spending minimum amount of money

Algebraic setup

Let x_1 x_2 x_3 x_4 denote dollars spent per issue:

Minimize $x_1 + x_2 + x_3 + x_4$

subject to

(1) $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50{,}000$

(2) $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100{,}000$

(3) $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25{,}000$

$x_1, x_2, x_3, x_4 \geq 0$ (can't unadvertise)

Optimum:

$$\left.\begin{aligned} x_1 &= 2050000/111 \\ x_2 &= 425000/111 \\ x_3 &= 0 \\ x_4 &= 625000/111 \end{aligned}\right\} \quad x_1 + x_2 + x_3 + x_4 = \frac{3\,100\,000}{111}$$

Linear Programming (LP)

- Minimize or maximize linear objective function subject to linear inequalities (& equations)
- variables $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$
- objective function: $\vec{c} \cdot \vec{x} = c_1 x_1 + c_2 x_2 + \cdots c_d x_d$
- inequalities: $A\vec{x} \leq \vec{b}$ ~ A is $n \times d$

  e.g., $x_1 - x_3 \leq 7$ $\quad (1\ 0\ {-1}\ 0\ 0)x \leq 7$
- thus: $\max\ \vec{c} \cdot \vec{x}$

  s.t. $A\vec{x} \leq \vec{b}$, $\vec{x} \geq 0$

CERTIFICATE OF OPTIMALITY

Is there a short certificate that shows LP solution is indeed optimal?

Consider $\frac{25}{222} \times (1) + \frac{46}{222} \times (2) + \frac{14}{222} \times (3)$

$$\Rightarrow x_1 + x_2 + \frac{140}{222} x_3 + x_4 \geq \frac{3100000}{111}$$

Since $x_1 + x_2 + x_3 + x_4 \geq x_1 + x_2 + \frac{140}{222} x_3 + x_4$

remember $x_i \geq 0$ no solution can be smaller than this

LP DUALITY

Short certificate is not a coincidence but a consequence of the following.

Theorem:

$$\begin{array}{ll} \max \vec{c} \cdot \vec{x} & \\ \text{s.t. } A\vec{x} \leq \vec{b} & \\ \vec{x} \geq 0 & \end{array} = \begin{array}{l} \min \vec{b} \cdot \vec{y} \\ \text{s.t. } A^T \vec{y} \geq \vec{c} \\ \vec{y} \geq 0 \end{array}$$

primal LP

dual LP

related to maxflow-mincut theorem

General algorithms

- Simplex algorithm: $\vec{x}$ walks from vertex to vertex in direction $\vec{c}$

  practical but worst-case exponential

- Ellipsoid algorithm: Guarantee OPT $\in$ ellipsoid
  shrink ellipsoid

  First poly time, useful in theory, impractical

- Interior point method: $\vec{x}$ moves inside polytope vaguely $\to \vec{c}$

  poly time & quite practical

CONVERTING TO STANDARD FORM

1) Want to minimize $-2x_1 + 3x_2$. Negate coefficients and maximize $2x_1 - 3x_2$.

2) If x_j does not have a non-negativity constraint. x_j replaced by $x_j' - x_j''$ $\quad x_j' \geq 0$, $x_j'' \geq 0$

3) Equality constraint $x_1 + x_2 = 7$ translates to
   $x_1 + x_2 \leq 7$, $x_1 + x_2 \geq 7$

4) $\geq$ constraint translated to $\leq$ by multiplication of -1
   $x_1 + x_2 \geq 7$
   to $-x_1 - x_2 \leq -7$

FORMULATING PROBLEMS AS LP

Difference constraints: $x_i - x_j \le w_{ij}$

special case of linear programming where each row of A has one +1 and one −1, & rest 0s

Solved by Bellman Ford

Maximum Flow

$$\max \sum_{v \in V} f(s,v) \quad = |f|$$

$$\text{s.t. } f(u,v) = -f(v,u) \quad \forall u,v \in V \quad \text{skew symmetry}$$

$$\sum_{v \in V} f(u,v) = 0 \quad \forall u \in V - \{s,t\} \quad \text{conservation}$$

$$f(u,v) \le c(u,v) \quad \forall u,v \in V \quad \text{capacity}$$

Shortest paths

From vertex s:

$$\max \sum_{V} d[v]$$

$$\text{s.t. } d[v] - d[u] \le w(u,v) \quad \forall (u,v) \in E$$

(triangle inequality)

$$d[s] = 0$$

max not a min

no solution $\Leftrightarrow$ neg-weight cycle reachable from s

SIMPLEX ALGORITHM

Works well in practice, but exponential in the worst case

Flow: Represent LP in slack form

Convert one slack form into an equivalent slack form where objective value has not decreased and has likely increased.

Keep going till the optimal solution becomes obvious

Think of Simplex as Gaussian Elimination on inequalities

Simplex Example

$$\text{Maximize} \quad 3x_1 + x_2 + x_3$$

subject to:

$$
\begin{aligned}
x_1 + x_2 + 3x_2 &\leq 30 \\
2x_1 + 2x_2 + 5x_3 &\leq 24 \\
4x_1 + x_2 + 2x_3 &\leq 36 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

nonbasic variables

Slack form:

$$
\left.
\begin{aligned}
z &= \;\; 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
\right\} \quad \mathrm{I}
$$

basic variables

Basic Solution: set all nonbasic variables on right hand side (r.h.s.), and compute the values of the basic variables on l.h.s.

Objective function: $3(0) + 1(0) + 1(0) = 0$

feasible solution, may not always be so!

Pivoting

1) Select a nonbasic variable x_e whose coefficient in the objective function is positive
2) Increase the value of x_e as much as possible w/o violating any of the constraints
3) Variable x_e becomes basic, some other variable becomes nonbasic (values of other basic variables & objective function may change)

Increase the value of x_1. 3rd constraint is the tightest one (-4 multiplier) and limits how much we can increase x_1.

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Rewrite other equations with x_6 on r.h.s. That is, replace x_1 with above equation's r.h.s.

$$\left.\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_1}{2}
\end{aligned}\right\} \text{II}$$

EQUIVALENCE

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$

Original basic solution: $(0, 0, 0, 30, 24, 36)$

satisfies $\underline{\mathrm{II}}$ and has objective value

$$27 + \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 0 - \frac{3}{4}\cdot 36 = 0$$

Basic solution for $\underline{\mathrm{II}}$: set nonbasic values to 0

$$(9, 0, 0, 21, 6, 0)$$

Basic Solution for $\underline{\mathrm{II}}$ satisfies I, objective value $= 27$

NEXT PIVOT

Increasing x_6 causes objective value to decrease $\left(-\frac{3x_6}{4}\right)$

x_2 or x_3 : choose x_3

Again 3rd constraint is the limiting factor

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$\left.\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11}{16}x_6 \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_2 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}\right\} \underline{\overline{\mathrm{III}}}$$

Basic solution $\underline{\text{III}}$: $\left(\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0\right)$

Objective value : $\frac{111}{4}$

Pivot on x_2

$$\left.\begin{aligned} z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\ x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\ x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\ x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} \end{aligned}\right\} \ \underline{\text{IV}}$$

all nonbasic variable coefficients in objective function are negative → reached optimal!

We won't prove that ...

constraints

Simplex converges in $\binom{n+m}{n}$ iterations

variables

Did not discuss:

How to determine if LP is feasible?

What if LP is feasible but the initial basic solution is infeasible?

How do we determine if the LP is unbounded?

How do we choose the pivot?