Strong Induction

Strong Induction

Prove P(0). Then prove P(n+1) assuming all of

P(0), P(1), …, P(n)

(instead of just P(n)).

Conclude $\forall m.P(m)$

Postage by Strong Induction

available stamps: 

5¢ 3¢

Thm: Get any amount $\geq$ 8¢

By strong induction with hyp:
P(n) ::= can form n + 8¢.

Postage by Strong Induction

available stamps: 

5¢ 3¢

Thm: Get any amount $\geq$ 8¢

base case P(0): make 0 + 8¢

Postage by Strong Induction

available stamps: 

5¢ 3¢

Thm: Get any amount $\geq$ 8¢

inductive step:

Assume m+8¢ for n $\geq$ m $\geq$ 0.

Postage by Strong Induction

available stamps: 

5¢ 3¢

Thm: Get any amount $\geq$ 8¢

inductive step:

Assume all from 8 to n+8¢.

Postage by Strong Induction

available stamps: 

5¢ 3¢

Thm: Get any amount $\geq$ 8¢

inductive step:

Assume all from 8 to n+8¢.

Prove can get n+9¢, for n$\geq$0

Postage by Strong Induction

inductive step cases:

n=0, 0+9¢ = 

n=1, 1+9¢ = 

Postage by Strong Induction

n $\geq$ 2, so by hypothesis
can get (n-2)+8¢

 + = n+9¢

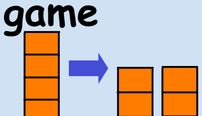
3¢

├─(n-2)+8¢─┤

Postage by Strong Induction

We conclude by strong induction that, using 3¢ and 5¢ stamps, n + 8¢ postage can be formed for all n $\geq$ 0.

Unstacking game



Start: a stack of boxes

Move: split any stack into two of sizes a,b>0

Scoring: a·b points

Keep moving: until stuck

Overall score: sum of move scores

Analyzing the Stacking Game

Claim: Every way of unstacking n blocks gives the same score:

$$(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$$

Analyzing the Game

Claim: Starting with size n stack, final score will be

$$\frac{n(n-1)}{2}$$

Proof: by Strong induction with Claim(n) as hypothesis

Proving the Claim by Induction

Base case $n = 0$:

$$\text{score} = 0 = \frac{0(0-1)}{2}$$

Claim(0) is O.K.

Proving the Claim by Induction

Inductive step. Assume for stacks $\leq n$, and prove $C(n+1)$:

$$(n+1)\text{-stack score} = \frac{(n+1)n}{2}$$

Proving the Claim by Induction

Inductive step.

Case $n+1 = 1$. verify for 1-stack:

$$\text{score} = 0 = \frac{1(1-1)}{2}$$

$C(1)$ is O.K.

Proving the Claim by Induction

Inductive step.

Case $n+1 > 1$. Split $n+1$ into an a-stack and b-stack, where $a + b = n+1$.

$(a + b)$-stack score $= ab +$ a-stack score $+$ b-stack score

Proving the Claim by Induction

by strong induction:

$$a\text{-stack score} = \frac{a(a-1)}{2}$$

$$b\text{-stack score} = \frac{b(b-1)}{2}$$

Proving the Claim by Induction

total (a + b)-stack score =

$$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} =$$

$$\frac{(a+b)((a+b)-1)}{2} = \frac{(n+1)n}{2}$$

so C(n+1) is O.K.

We're done!