In-Class Problems Week 6, Mon.

Problem 1.
Find

$$\text{remainder}\left(9876^{3456789}\left(9^{99}\right)^{5555} - 6789^{3414259},\ 14\right). \tag{1}$$

Problem 2.
Suppose a, b are relatively prime and greater than 1. In this problem you will prove the *Chinese Remainder Theorem*, which says that for all m, n, there is an x such that

$$x \equiv m \bmod a, \tag{2}$$
$$x \equiv n \bmod b. \tag{3}$$

Moreover, x is unique up to congruence modulo ab, namely, if x' also satisfies (2) and (3), then

$$x' \equiv x \bmod ab.$$

(a) Prove that for any m, n, there is some x satisfying (2) and (3).

Hint: Let b^{-1} be an inverse of b modulo a and define $e_a ::= b^{-1}b$. Define e_b similarly. Let $x = me_a + ne_b$.

(b) Prove that

$$[x \equiv 0 \bmod a \ \text{ AND } \ x \equiv 0 \bmod b] \quad \text{implies} \quad x \equiv 0 \bmod ab.$$

(c) Conclude that

$$\left[x \equiv x' \bmod a \ \text{ AND } \ x \equiv x' \bmod b\right] \quad \text{implies} \quad x \equiv x' \bmod ab.$$

(d) Conclude that the Chinese Remainder Theorem is true.

(e) What about the converse of the implication in part (c)?

Problem 3.

Definition. The set, P, of integer polynomials can be defined recursively:

Base cases:

- the identity function, $\text{Id}_{\mathbb{Z}}(x) ::= x$ is in P.
- for any integer, m, the constant function, $c_m(x) ::= m$ is in P.

Constructor cases. If $r, s \in P$, then $r + s$ and $r \cdot s \in P$.

(a) Using the recursive definition of integer polynomials given above, prove by structural induction that for all $q \in P$,

$$j \equiv k \pmod{n} \quad \text{IMPLIES} \quad q(j) \equiv q(k) \pmod{n},$$

for all integers j, k, n where $n > 1$.

Be sure to clearly state and label your Induction Hypothesis, Base case(s), and Constructor step.

(b) We'll say that q *produces multiples* if, for every integer greater than one in the range of q, there are infinitely many different multiples of that integer in the range. For example, if $q(4) = 7$ and q produces multiples, then there are infinitely many different multiples of 7 in the range of q.

Prove that if q has positive degree and positive leading coefficient, then q produces multiples. You may assume that every such polynomial is strictly increasing for large arguments.

Hint: Observe that all the elements in the sequence

$$q(k), q(k + v), q(k + 2v), q(k + 3v), \ldots,$$

are congruent modulo v. Let $v = q(k)$.