ALBERT MEYER: We've considered arithmetic sums, where each term is a fixed amount larger than the previous term by an additive amount. Another kind of sum called geometric sums which, we're going to look at now, come about where each sum is a fixed multiple of the previous sum. And these come up all the time in lots of different settings. And in this particular case, we're going to look at an example of how it applies to analyzing the value of money in the future.

So let's begin with a geometric sum. And there's the standard form of it. Geometric sum is of the form 1 plus x plus x squared up through the nth power of x. And, for uniformity, notice that 1 is actually x to the 0. So we're taking the sum from k equals 0 to n of x to the k. Now, what I'd like to do is find a nice closed form for this without those ellipses, and having a growing number of terms n. And there's a simple trick with the arithmetic sum.

The way Gauss got it and the way we got it was by reversing the sum and adding the things together. This time the trick is to multiply Gn by x. Now what's that going to do is it's going to increase the power of x in each term by one, which is tantamount to a right shift. Let's look at that. There's xG of n, so 1 times x, course, I'm subtracting looking ahead.

So 1 times x is minus x times x is x squared. I'm going to subtract it down through x to the n minus 1 times x gives me the x to the nth term. And finally, I have an extra term from right shifting x to the n to be x to the n plus 1. Now let's do this subtraction. But of course, I'm going to align the terms up so that they're easy to subtract. And now all the terms in the middle cancel, which is very cool, because we've just figured out that Gn minus xGn is 1 minus x to the n plus 1.

So we have this nice, elegant formula. Now I can factor out Gn and I get a Gn times 1 minus x on the left. And the result is that I get the Gn is 1 minus x to the n plus 1 over 1 minus x. This is actually a formula that we proved before by induction. But when we did it by induction, there was no clue about who was the clever person to think of this formula. Now you know how that clever person found it.

And this is kind of a standard trick that we'll see more of when we look at generating functions. But for now, it's a simple trick for getting a nice closed form for a sum. We refer to it as the perturbation method. You take the sum, you perturb it a little, see how it relates to itself, get an arithmetic relation, and solve for a formula for the sum. OK.

A geometric series-- I use the word sum for a finite sum-- a geometric series is when you take an infinite sum. So the infinite geometric sum is the sum 1 plus x plus x squared x to the n. And it keeps going. It's the sum from i equals 0 to infinity of x to the i. And there's a simple formula for that too. It's even simpler actually.

Because the definition of an infinite sum is it's the limit of the truncated sums. It's the limit of the sum of the first n terms as n goes to infinity, assuming that limit exists. And so for the value of this infinite series is the limit of Gn, which is the sum up to n, and let's look at that. Well, Gn is 1 minus x to the n plus 1 over 1 minus x.

So the limit of that, the limits distribute pass the 1 down to-- and the x doesn't have an n in it, so it winds up being 1 minus the limit as n approaches infinity of x to the n plus 1 divided by 1 minus x. And as long as x is less than 1, x to the n plus 1 is going to go to 0, and I wind up with this nice simple formula that the infinite series, the sum from i equals 0 to infinity of x to the i is equal to 1 over 1 minus x, providing that the magnitude of x is less than 1.

OK. That's the basics mathematical preliminaries of geometric sums and geometric series. Now, let's look at a typical application having to do with the future value of money. Suppose we want to make the following deal, I promise I will pay you $100 in one year if you will pay me a fixed amount now. So let's call it x dollars.

And the puzzle is how much money is $100 worth if you can't have it now? You can only have it in one year. It's worth x dollars. How are we going to figure out what x should be? What would be a fair amount for you to pay me so that I'm willing to pay you $100 in one year and it's fair, nobody loses?

OK. Well, here's the basic fact that is the basis for evaluating what the value of money in the future is is I'm going to assume that my bank will pay me 3% interest. This is a generous bank in today's economy, but it used to be a stingy offer. Interest rates in my lifetime have ranged between about 17% a year down to less than 1% a year. 3% is a reasonable number to play with.

So let's suppose that my bank commits to paying me 3% interest on a deposit now. That is to say, let's define the bank rate, b, to be 1.03. And the deal is that the bank will increase the money that I have now by a factor of b in one year. OK. Well, so if I deposit your x dollars now, that means I will have b times x dollars in one year. OK.

Assuming that the bank is completely reliable, there's no risk there and I get exactly b times x in one year, then I won't lose any money providing that b times x is greater than or equal to 100. I need the x dollars you give me now to be worth the $100 I'm supposed to pay you. I'll come out ahead if bx is greater than or equal to 100. I'll loose if it's less than 100. And it's completely fair if bx is equal to 100. All right. Well, that means that x is simply 100 over b, which we decided was 1.03 $97.09. So $100 in one year is worth $97.09 or normalizing to $1.00, $1.00 in one year is worth $0.97 essentially now.

Well, now we can shift perspective a little bit and think back a year. So how much money did I need last year in order to be worth $1.00 today? Well, by the same reasoning, the bank is going to pay me b times r today. So I need b times r to equal a dollar. In other words, r has got to be 1 over , r is 1 over the bank rate. So $1.00 a year ago is worth-- r dollars a year ago is worth a dollar today.

And by the same reasoning, n dollars paid in two years is worth n times r paid in one year, which is worth n times r squared paid today. So I can iterate this one over bank rate factor, and as time goes on, k years out-- a value of n dollars paid in k years is worth n times r to the k today, where r is 1 over the bank rate. OK. That's good to know.

Let's think about annuities now. An annuity is a contract that people by to provide income for themselves without risk. So they will make a deal typically with an insurance company where they will pay a certain amount of money now to the insurance company, and the insurance company promises to provide them a regular income sometimes for life or sometimes for a fixed period. So let's look at an example.

I will pay you $100 a year for 10 years if you will pay me a fixed premium. What should it be? So I'm going to promise as the insurance company to pay you $100 a year for 10 years. I want you to pay me a premium of y dollars now. How much should you pay? Well, let's think about it.

$100 in one year is worth 100 times r. And $100 in two years is worth 100 times r squared. And finally, $100 in 10 years is worth 100 times r to the 10th. So this is the amount that I will have to pay you in today's dollars-- I need-- in order to be paying you $100 a year for 10 years. I need a total of this much amount of money now, because each of these terms is the amount of money that the $100 will be worth paid to you that many years out.

Well, look at this sum. If I factor out 100 r, I'm left with 100 r times the geometric sum from 1 to r to the ninth, where the base of the sum is r, the factor is r. Well, we have a nice formula for that. It's simply 1 minus r to the 10th over 1 minus r. And now plugging in r equals 1 over 1.03. I wind up with the conclusion that this annuity is worth $853.02 today.

My promise to pay you $1,000, but spread out over the next 11 years, is worth $853.02 today, assuming that the bank rate is 3% a year. And that's a typical case where geometric series come up. And you'll see other examples in problems.

Just a quick thing to think about. Suppose that the bank rates rapidly increase, unexpectedly increase. The Fed finally gets the economy moving and interest rates run up to 5%, say. What happens on this annuity? You've already paid me the 853 and I've already committed to paying you $100 a year for the next 10 years starting in a year. Who comes out ahead if bank rates increase? You come out ahead, the deal stays fair, or I come out ahead. And I'll close by letting you think about that question.