Problem Set 12

Problem 1. [15 points]

In this problem, we will (hopefully) be making tons of money! Use your knowledge of probability and statistics to keep from going broke!

Suppose the stock market contains N types of stocks, which can be modelled by independent random variables. Suppose furthermore that the behavior of these stocks is modelled by a double-or-nothing coin flip. That is, stock S_i has half probability of doubling its value and half probability of going to 0. The stocks all cost a dollar, and you have N dollars. Say you only keep these stocks for one time-step (that is, at the end of this timestep, all stocks would have doubled in value or gone to 0).

(a) [3 pts] What is your expected amount of money if you spend all your money on one stock? Your variance?

(b) [3 pts] Suppose instead you diversified your purchases and bought N shares of all different stocks. What is your expected amount of money then? Your variance?

(c) [3 pts] The money that you have invested came from your financially conservative mother. As a result, your goals are much aligned with hers. Given this, which investment strategy should you take?

(d) [3 pts] Now instead say that you make money on rolls of dice. Specifically, you play a game where you roll a standard six-sided dice, and get paid an amount (in dollars) equal to the number that comes up. What is your expected payoff? What is the variance?

(e) [3 pts] We change the rules of the game so that your payoff is the cube of the number that comes up. In that case, what is your expected payoff? What is its variance?

Problem 2. [10 points] Here are seven propositions:

$$
\begin{array}{ccccc}
x_1 & \vee & x_3 & \vee & \neg x_7 \\
\neg x_5 & \vee & x_6 & \vee & x_7 \\
x_2 & \vee & \neg x_4 & \vee & x_6 \\
\neg x_4 & \vee & x_5 & \vee & \neg x_7 \\
x_3 & \vee & \neg x_5 & \vee & \neg x_8 \\
x_9 & \vee & \neg x_8 & \vee & x_2 \\
\neg x_3 & \vee & x_9 & \vee & x_4
\end{array}
$$

Note that:

1. Each proposition is the OR of three terms of the form x_i or the form $\neg x_i$.

2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

(a) [5 pts] What is the expected number of true propositions?

(b) [5 pts] Use your answer to prove that there exists an assignment to the variables that makes *all* of the propositions true.

Problem 3. [20 points] MIT students sometimes delay laundry for a few days (to the chagrin of their roommates). Assume all random variables described below are mutually independent.

(a) [5 pts] A ***busy*** student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability 2/3 and 2 days with probability 1/3. Let B be the number of days a busy student delays laundry. What is $\mathrm{E}[B]$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

(b) [5 pts] A ***relaxed*** student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\mathrm{E}[R]$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

(c) [5 pts] Before doing laundry, an ***unlucky*** student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\mathrm{E}[U]$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

(d) [5 pts] A student is ***busy*** with probability 1/2, ***relaxed*** with probability 1/3, and ***unlucky*** with probability 1/6. Let D be the number of days the student delays laundry. What is $\mathrm{E}[D]$?

Problem 4. [10 points] We have two coins: one is a fair coin and the other is a coin that produces heads with probability 3/4. One of the two coins is picked, and this coin is tossed n times. Explain how to calculate the number of tosses to make us 95% confident which coin was chosen. You do not have to calculate the minimum value of n, though we'd be pleased if you did.

Problem 5. [13 points] Each 6.042 final exam (out of 100 points) will be graded according to a rigorous procedure:

- With probability $\frac{4}{7}$ the exam is graded by a *TA*; with probability $\frac{2}{7}$ it is graded by a *lecturer*; and with probability $\frac{1}{7}$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.

- *TAs* score an exam by scoring each problem individually and then taking the sum.

  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability 3/4, and no credit is given with probability 1/4.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.

- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a "general impression" score.

  - With probability $\frac{4}{10}$, the general impression score is 40.
  - With probability $\frac{3}{10}$, the general impression score is 50.
  - With probability $\frac{3}{10}$, the general impression score is 60.

Assume all random choices during the grading process are independent.

(a) [5 pts] What is the expected score on an exam graded by a TA?

(b) [5 pts] What is the expected score on an exam graded by a lecturer?

(c) [3 pts] What is the expected score on a 6.042 final exam?

Problem 6. [32 points]

Suppose n balls are thrown randomly into n boxes, so each ball lands in each box with uniform probability. Also, suppose the outcome of each throw is independent of all the other throws.

(a) [5 pts] Let X_i be an indicator random variable whose value is 1 if box i is empty and 0 otherwise. Write a simple closed form expression for the probability distribution of X_i. Are $X_1, X_2, \ldots, X_n$ independent random variables?

(b) [5 pts] Find a constant, c, such that the expected number of empty boxes is asymptotically equal ($\sim$) to cn.

(c) [5 pts] Show that

$$\text{Pr (at least } k \text{ balls fall in the first box)} \leq \binom{n}{k}\left(\frac{1}{n}\right)^k.$$

(d) [7 pts] Let R be the maximum of the numbers of balls that land in each of the boxes. Conclude from the previous parts that

$$\Pr\{R \geq k\} \leq \frac{n}{k!}.$$

(e) [10 pts] Conclude that
$$\lim_{n\to\infty} \Pr\{R \geq n^\epsilon\} = 0$$

for all $\epsilon > 0$.

6.042J / 18.062J Mathematics for Computer Science
Fall 2010