Sums & Products

C. F. Gauss

Picture source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Gauss.html

Sum for Children

$$\begin{array}{lllr} 89 & + 102 + 115 + 128 + 141 & & + \\ 154 & + & \cdots & + \\ 193 & + & \cdots & + \\ 232 & + & \cdots & + \\ 323 & + & \cdots & + \\ 414 & + & \cdots + 453 + 466 & \end{array}$$

Sum for Children

Nine-year old Gauss saw

30 numbers each 13 greater than the previous one.

(So the story goes.)

Sum for Children

$1^{st} + 30^{th} = 89+466 \qquad = 555$

$2^{nd} + 29^{th} =$

$\quad (1^{st}+13) + (30^{th}-13) \qquad = 555$

$3^{rd} + 28^{th} =$

$\quad (2^{nd}+13) + (29^{th}-13) \qquad = 555$

Sum for Children

Sum of k^{th} term and $(31-k)^{th}$ term is invariant!

Total $= 555 \cdot 15$

$\quad = (1^{st} + \text{last}) \cdot (\#\text{ terms}/2)$

$\quad = ((1^{st} + \text{last})/2) \cdot (\#\text{ terms})$

Sum for Children

Example:

$$1 + 2 + \cdots + (n-1) + n = \frac{(1+n)n}{2}$$

 October 26, 2005 L8-2.7

Geometric Series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$
$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$

 October 26, 2005 L8-2.8

Geometric Series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$
$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$
$$G_n - xG_n = 1 - x^{n+1}$$

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Geometric Series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$
$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$
$$G_n - xG_n = 1 - x^{n+1}$$

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

 October 26, 2005 L8-2.10

Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Consider the *infinite* sum (series)

$$1 + x + x^2 + \cdots + x^{n-1} + x^n + \cdots = \sum_{i=0}^{\infty} x^i$$

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Infinite Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

$$\lim_{n\to\infty} G_n = \frac{1 - \lim_{n\to\infty} x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

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Infinite Geometric Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $x < 1$

Team Problem

Problem 1

The future value of $$

I will promise to pay you $100
in exactly one year,
if you will pay me $X now.

The future value of $$

My bank will pay me 3% interest.

Define *bankrate:*

$$b ::= 1.03$$

-- the factor by which bank will increase my holdings in 1 year.

The future value of $$

If I deposit your $X for a year,
I will have $$(b \cdot X)$.
So I won't lose money as long as

$$bX \geq 100.$$

$$X \geq \$100/1.03 \approx \$97.09$$

The future value of $$

- $1 in a year is worth $ 0.9709 today
- $$n$ is worth $$nr$ a year earlier, where $r ::= 1/b$.
- So $$n$ paid in two years is worth $$nr$ paid in one year, and is worth $$nr^2$ today.

The future value of $$

$n paid *k* years from now

is worth $$nr^k$ today

where r ::= 1/bankrate.

Annuities

I will pay you $100/year for 10 years

if you will pay me $Y now.

I can't lose if you pay me

$100r + 100r^2 + 100r^3 + \ldots + 100r^{10}$

$= 100r(1 + r + \ldots + r^9)$

$= 100r(1-r^{10})/(1-r) = \$853.02$

Annuities

I will pay you $100/year for 10 years

if you will pay me $853.02 now.

QUICKIE: If bankrates unexpectedly increase in the next few years,

A. I come out ahead

B. You come out ahead

C. The deal stays fair

Manipulating Sums

$$\frac{d}{dx} \quad \sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=0}^{n} ix^{i-1} = \frac{1}{x}\sum_{i=1}^{n} ix^i = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right)$$

Manipulating Sums

$$\sum_{i=1}^{n} ix^i = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Team Problem

Problems 2,3