Mathematics for Computer Science

MIT 6.042J/18.062J

Planar Graphs; Bipartite Matching

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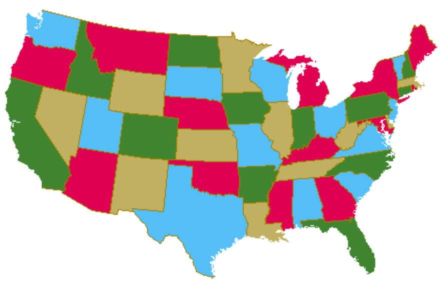
Planar Graphs

Planar Graphs



Planar Graphs

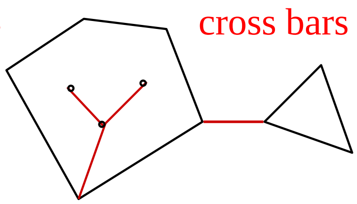
A graph is ***planar*** if there is a way to draw it in the plane without edges crossing.

Planar Graphs

Maps are 2-connected planar graphs

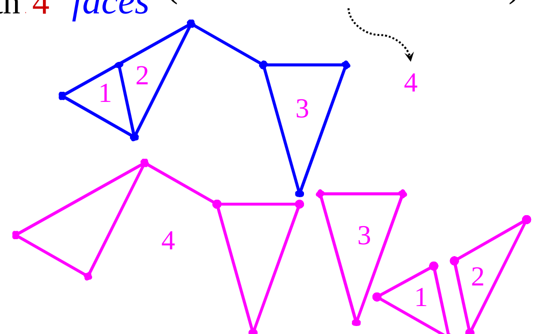
General connected planar graphs may have

dongles cross bars



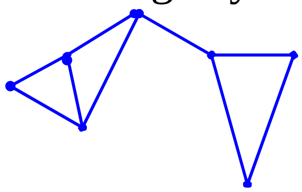
A Planar Graph

with 4 *faces* (wait! also the outer face)



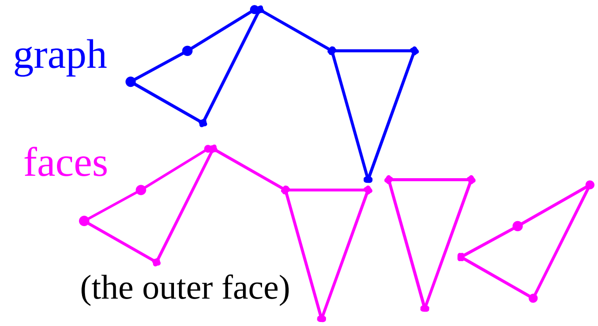
Planar Graphs

draw it edge by edge:



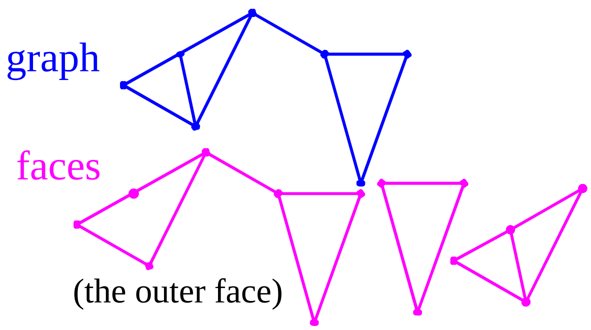
Planar Graphs

and record faces while drawing



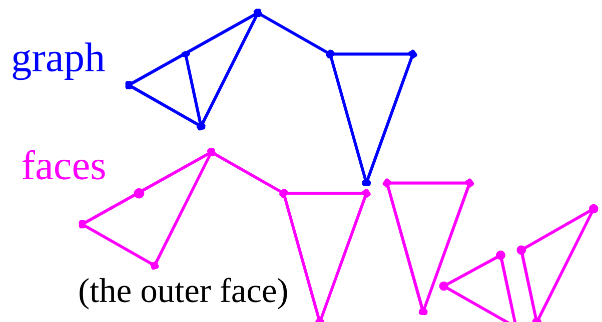
Planar Graphs

and record faces while drawing



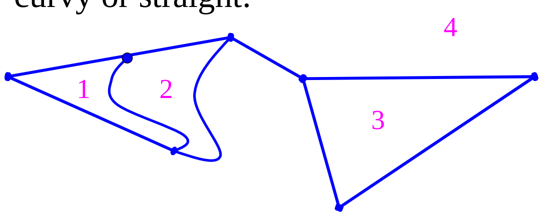
Planar Graphs

and record faces while drawing



If you like curves…

With same faces, you can draw the graph in the plane big or small, curvy or straight:



"Planar Drawing" = Faces

An (abstract) *planar drawing* is defined to be its set of faces.

The same planar graph may have different drawings.

Team Problem

Problem 1

Euler's Formula

If a *connected* planar drawing has v vertices, e edges, and f faces, then

$$v - e + f = 2$$

Euler's Formula

Proof by induction on # edges in drawing:

base case: no edges

connected, so $v = 1$ ●
outside face only, so $f = 1$
$e = 0$

$$1 - 0 + 1 = 2$$ O.K.

Adding an edge to a drawing

Inductive step: any n+1 edge drawing comes from adding an edge to some n edge drawing.

(not a buildup error: it's the *definition* of drawing edge by edge)

So can assume Euler for n edge drawing and see what happens to $v - e + f$ when 1 edge is added.

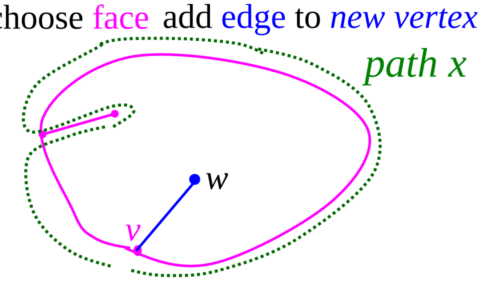
Adding an edge to a drawing

Two cases for connected graph:

1) Attach edge from vertex on a face to a new vertex.
2) Attach edge between vertices on a face.

Face Creation Rules

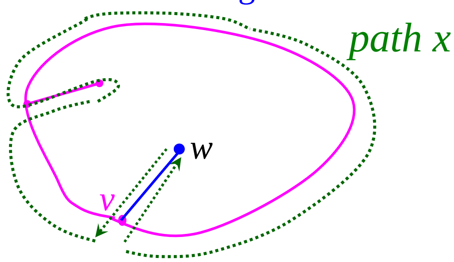
1) choose face add edge to *new vertex*



old face ~~vx~~v

Face Creation Rules

1) choose face add edge to *new vertex*



new face is *wvxvw*

Face Creation Rules

1) choose face add edge to *new vertex*

nothing else changes

new face is *wvxvw*

Euler's Formula

v increases by 1

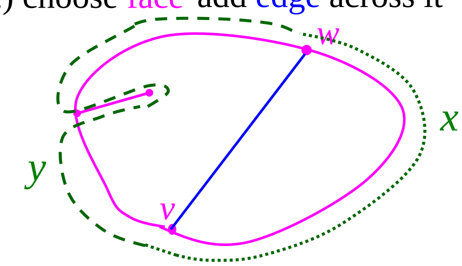
e increases by 1

f stays the same

so *v–e+f* stays the same

Face Creation Rules

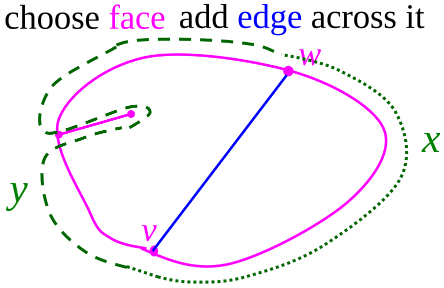
2) choose face add edge across it



old face: *wx~~v~~yw*

Face Creation Rules

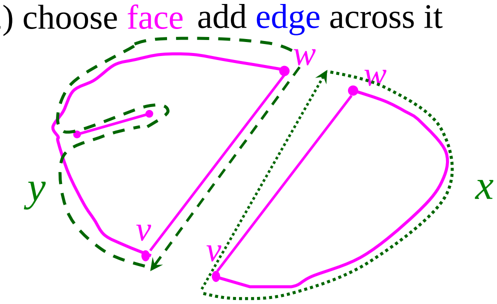
2) choose face add edge across it



splits into 2 faces:

Face Creation Rules

2) choose face add edge across it



splits into 2 faces: *wxvw*, *vywv*

Face Creation Rules

2) choose face add edge across it

nothing else changes

splits into 2 faces: *wxvw*, *vywv*

Euler's Formula

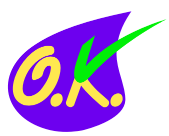
v stays the same

e increases by 1

f increases by 1

so $v-e+f$ stays the same

Euler's Formula

Inductive step: 

Team Problems

Problems

2 & 3

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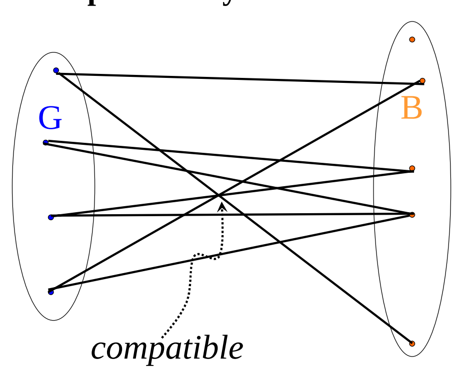
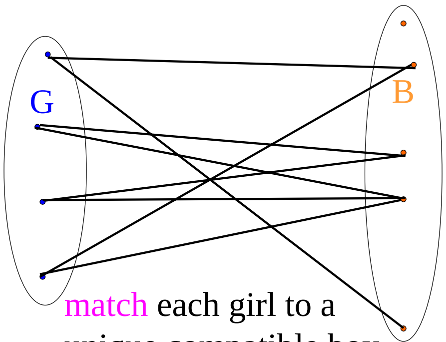
Bipartite Matching: Hall's Theorem

Compatible Boys & Girls



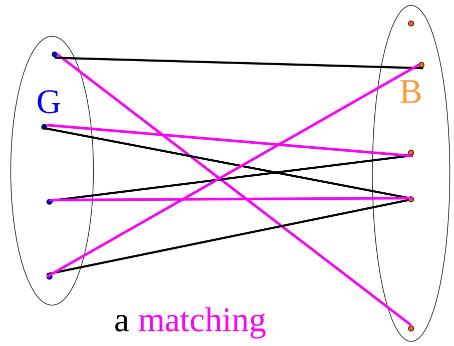
Compatible Boys & Girls



match each girl to a unique compatible boy

 October 7, 2005 lec 5f.31

Compatible Boys & Girls



a matching

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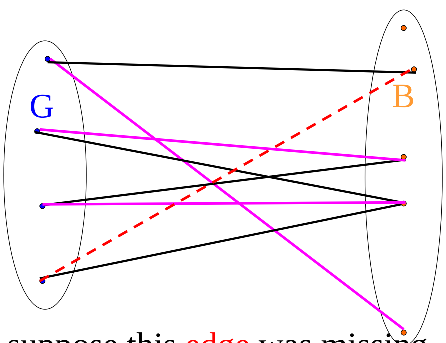
Compatible Boys & Girls



suppose this edge was missing

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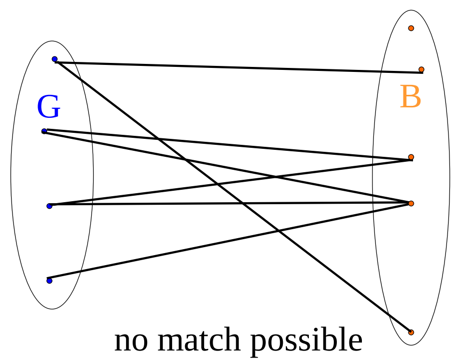
Compatible Boys & Girls



no match possible

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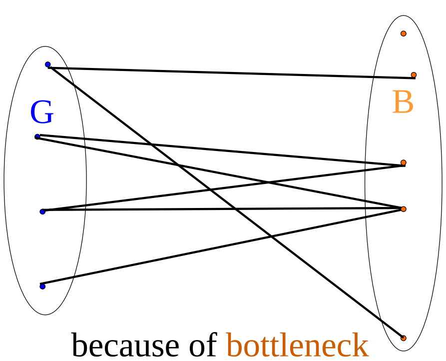
No match possible



because of bottleneck

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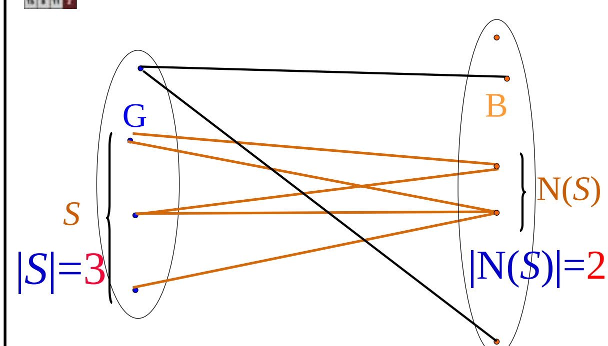
Bottleneck condition



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Bottleneck Lemma

bottleneck: not enough boys for some set of girls.

If there is a bottleneck, then no match is possible.

$S \subseteq G,\ \mathrm{N}(S) ::= \{b \mid b \text{ adjacent to a } g \in S\},$

$|S| > |\mathrm{N}(S)|$

Hall's Theorem

There is a perfect match iff there are no bottlenecks.

Proof in Notes: clever strong induction on #girls.

(Better proof using *duality principle* goes beyond 6.042)

Hall's Theorem

There is a perfect match iff there are no bottlenecks.

Lots of elegant use in applications & proofs

Team Problem

Problem 4