Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Fall 2010)

**Problem Set 5: Solutions**

1. (a) Because of the required normalization property of any joint PDF,

$$1 = \int_{x=1}^{2}\left(\int_{y=x}^{2} ax\ dy\right) dx \ = \ \int_{x=1}^{2} ax(2-x)\ dx \ = \ a\left(2^2 - 1^2 - \frac{2^3}{3} + \frac{1^3}{3}\right) \ = \ \frac{2}{3}a$$

so $a = 3/2$.

(b) For $1 \le y \le 2$,

$$f_Y(y) = \int_1^y ax\ dx \ = \ \frac{a}{2}(y^2-1) \ = \ \frac{3}{4}(y^2-1),$$

and $f_Y(y) = 0$ otherwise.

(c) First notice that for $1 \le x \le 3/2$,

$$f_{X|Y}(x \mid 3/2) = \frac{f_{X,Y}(x,3/2)}{f_Y(3/2)} = \frac{(3/2)x}{\frac{3}{4}\left(\left(\frac{3}{2}\right)^2 - 1^2\right)} = \frac{8x}{5}.$$

Therefore,

$$\mathbf{E}[1/X \mid Y = 3/2] \ = \ \int_1^{3/2} \frac{1}{x}\frac{8x}{5}\,dx \ = \ 4/5.$$

2. (a) By definition $f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y \mid x)$. $f_X(x) = ax$ as shown in the graph. We have that

$$1 \ = \ \int_0^{40} ax\,dx \ = \ 800a.$$

So $f_X(x) = x/800$. From the problem statement $f_{Y|X}(y \mid x) = \frac{1}{2x}$ for $y \in [0, 2x]$. Therefore,

$$f_{X,Y}(x,y) = \begin{cases} 1/1600, & \text{if } 0 \le x \le 4 \text{ and } 0 < y < 2x, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Paul makes a positive profit if $Y > X$. This occurs with probability

$$\mathbf{P}(Y > X) \ = \ \int\int_{y>x} f_{X,Y}(x,y)\,dy\,dx \ = \ \int_0^{40}\int_x^{2x} \frac{1}{1600}\,dy\,dx \ = \ \frac{1}{2}.$$

We could have also arrived at this answer by realizing that for each possible value of $X$, there is a 1/2 probability that $Y > X$.

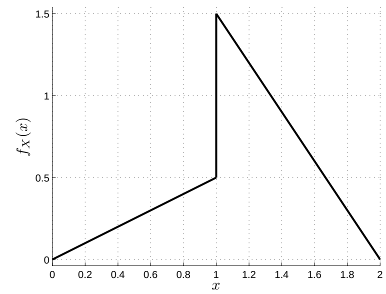
(c) The joint density function satisfies $f_{X,Z}(x,z) = f_X(x)\,f_{Z|X}(z|x)$. Since $Z$ is conditionally uniformly distributed given $X$, $f_{Z|X}(z \mid x) = \frac{1}{2x}$ for $-x \le z \le x$. Therefore, $f_{X,Z}(x,z) = 1/1600$ for $0 \le x \le 40$ and $-x \le z \le x$. The marginal density $f_z(z)$ is calculated as

$$f_Z(z) \ = \ \int_x f_{X,Z}(x,z)\,dx \ = \ \int_{x=|z|}^{40} \frac{1}{1600}\,dx \ = \ \begin{cases} \frac{40-|z|}{1600}, & \text{if } |z| < 40, \\ 0, & \text{otherwise.} \end{cases}$$

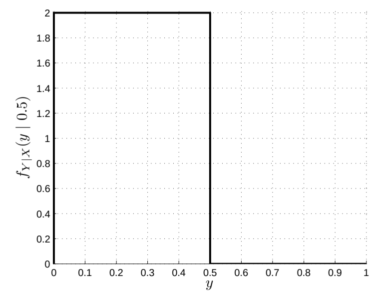
3. (a) In order for $X$ and $Y$ to be independent, any observation of $X$ should not give any information on $Y$. If $X$ is observed to be equal to 0, then $Y$ must be 0.

   In other words, $f_{Y|\{X=0\}}(y \mid 0) \neq f_Y(y)$. Therefore, $X$ and $Y$ are not independent.

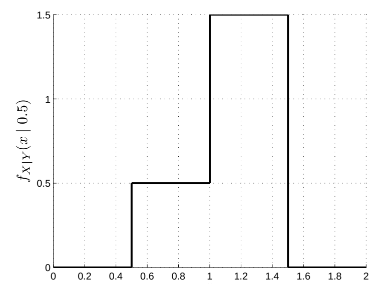
   (b) $f_X(x) = \begin{cases} x/2, & \text{if } 0 \le x \le 1, \\ -3x/2 + 3, & \text{if } 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$




   $f_{Y|X}(y \mid 0.5) = \begin{cases} 2, & \text{if } 0 \le y \le 1/2, \\ 0, & \text{otherwise.} \end{cases}$



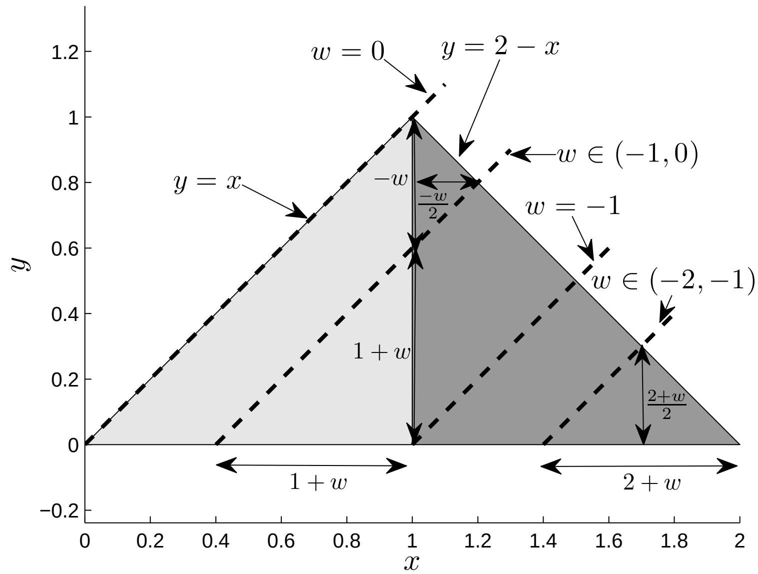

   $f_{X|Y}(x \mid 0.5) = \begin{cases} 1/2, & \text{if } 1/2 \le x \le 1, \\ 3/2, & \text{if } 1 < x \le 3/2, \\ 0, & \text{otherwise.} \end{cases}$




   (c) The event $A$ leaves us with a right triangle with a constant height. The conditional PDF is then $1/\text{area} = 8$. The conditional expectation yields:

$$\begin{aligned} \mathbf{E}[R \mid A] &= \mathbf{E}[XY \mid A] \\ &= \int_0^{0.5} \int_y^{0.5} 8xy \; dx \; dy \\ &= 1/16. \end{aligned}$$

   (d) The CDF of $W$ is $F_W(w) = \mathbf{P}(W \le w) = \mathbf{P}(Y - X \le w) = \mathbf{P}(Y \le X + w)$. $\mathbf{P}(Y \le X + w)$ can be computed by integrating the area below the line $Y = X + w$ for all possible values of $w$. The lines $Y = X + w$ are shown below for $w = 0$, $w = -1/2$, $w = -1$ and $w = -3/2$. The probabilities of interest can be calculated by taking advantage of the uniform PDF over the two triangles. Remember to multiply the areas by the appropriate joint density $f_{X,Y}(x, y)$! Take note that there are 4 regions of interest: $w < -2$, $-2 \le w \le -1$, $-1 < w \le 0$ and $w > 0$.



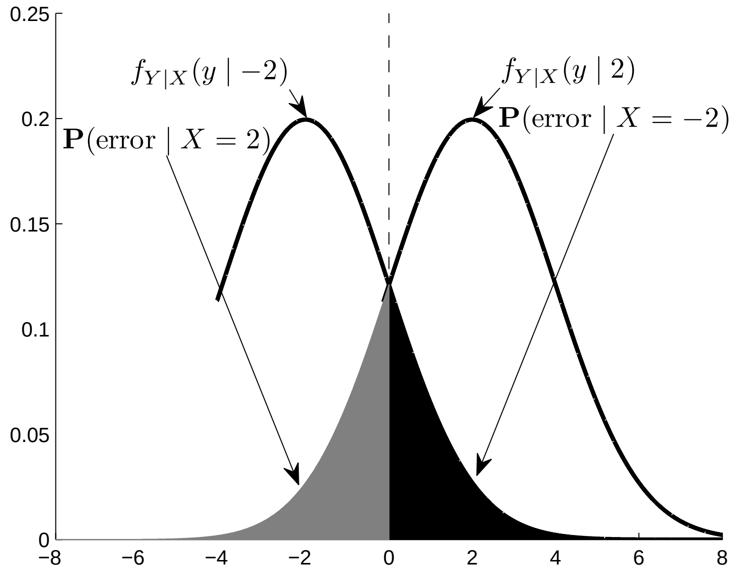

The CDF of $W$ is

$$F_W(w) = \begin{cases} 0, & \text{if } w < -2, \\ 3/2 \cdot 1/2(2+w)^2/2, & \text{if } -2 \le w \le -1, \\ 1/2 \cdot 1/2(1+w)^2 + 3/2 \cdot (1/2 \cdot 1 \cdot 1 - 1/2(-w/2 \cdot -w)), & \text{if } -1 < w \le 0, \\ 1, & \text{if } w > 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } w < -2, \\ 3/8 \cdot (2+w)^2, & \text{if } -2 \le w \le -1, \\ 1/8 \cdot (-w^2 + 4w + 8), & \text{if } -1 < w \le 0, \\ 1, & \text{if } w > 0. \end{cases}$$

As a sanity check, $F_W(-\infty) = 0$ and $F_W(+\infty) = 1$. Also, $F_W(w)$ is continuous at $w = -2$ and at $w = -1$.

4. (a) If the transmitter sends the 0 symbol, the received signal is a normal random variable with a mean of $-2$ and a variance of 4. In other words, $f_{Y|X}(y \mid -2) = \mathcal{N}(-2, 4)$.
Also, $f_{Y|X}(y \mid 2) = \mathcal{N}(2, 4)$ These conditional pdfs are shown in the graph below.




The probability of error can be found using the total probability theorem.

$$\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid X = -2)\mathbf{P}(X = -2) + \mathbf{P}(\text{error} \mid X = 2)\mathbf{P}(X = 2) \\
&= \frac{1}{2}(\mathbf{P}(Y \geq 0 \mid X = -2) + \mathbf{P}(Y < 0 \mid X = 2)) \\
&= \frac{1}{2}(\mathbf{P}(N \geq 2 \mid X = -2) + \mathbf{P}(N < -2 \mid X = 2)) \\
&= \frac{1}{2}(\mathbf{P}(N \geq 2) + \mathbf{P}(N < -2)) \\
&= \frac{1}{2}(\mathbf{P}(\frac{N-0}{2} \geq \frac{2-0}{2}) + \mathbf{P}(\frac{N-0}{2} < \frac{-2-0}{2})) \\
&= \frac{1}{2}((1 - \Phi(1)) + (1 - \Phi(1))) \\
&= 0.1587.
\end{aligned}$$

(b) With 3 components, the probability of error given an obervation of $X$ is the probability of decoding 2 or 3 of the components incorrectly. For each component, the probability of error is 0.1587. Therefore,

$$\begin{aligned}
\mathbf{P}(\text{error} \mid \text{sent } 0) &= \binom{3}{2}(0.1587)^2(1 - 0.1587) + (0.1587)^3 \\
&= 0.0676.
\end{aligned}$$

By symmetry, $\mathbf{P}(\text{error} \mid \text{sent } 1) = \mathbf{P}(\text{error} \mid \text{sent } 0)$.
Therefore, $\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \text{sent } 0)\mathbf{P}(\text{sent } 0) + \mathbf{P}(\text{error} \mid \text{sent } 1)\mathbf{P}(\text{sent } 1) = 0.0676$.

5. (a) There are many ways to show that $X$ and $Y$ are not independent. One of the most intuitive arguments is that knowing the value of $X$ limits the range of $Y$, and vice versa. For instance, if it is known in a particular trial that $X \geq 1/2$, the value of $Y$ in that trial cannot be smaller

than 1/2. Another way to prove that the two are not independent is to calculate the product of their expectations, and show that this is not equal to $\mathbf{E}[XY]$.

(b) Applying the definition of a marginal PDF,

for $0 \leq x \leq 1$,

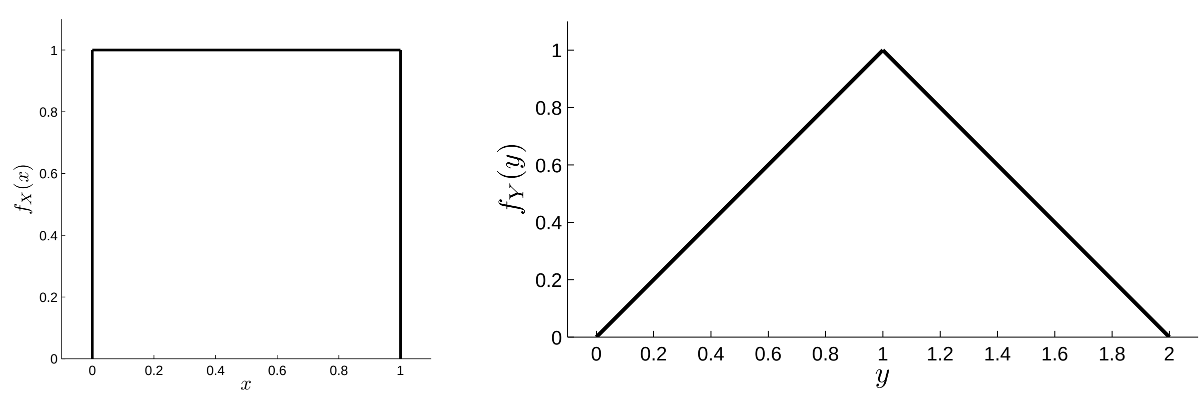
$$\begin{aligned} f_X(x) &= \int_y f_{X,Y}(x,y)\ dy \\ &= \int_x^{x+1} 1\ dy \\ &= 1; \end{aligned}$$

for $0 \leq y \leq 1$,

$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\ dx \\ &= \int_0^{y} 1\ dx \\ &= y; \end{aligned}$$

and for $1 \leq y \leq 2$,

$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\ dx \\ &= \int_{y-1}^{1} 1\ dx \\ &= 2-y. \end{aligned}$$




(c) By linearity of expectation, the expected value of a sum is the sum of the expected values. By inspection, $\mathbf{E}[X] = 1/2$ and $\mathbf{E}[Y] = 1$.
Thus, $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] = 3/2$.

(d) The variance of $X + Y$ is

$$\mathbf{E}[(X+Y)^2] - \mathbf{E}[X+Y]^2 = \mathbf{E}[X^2] + 2\mathbf{E}[XY] + \mathbf{E}[Y^2] - (\mathbf{E}[X+Y])^2. \tag{1}$$

In part (c), $\mathbf{E}[X+Y]$ was computed, so only the other three expressions need to be calculated. First, the expected value of $X^2$:

$$\mathbf{E}[X^2] \;=\; \int_0^1 x^2 \int_x^{x+1} 1\; dy\; dx \;=\; \int_0^1 x^2\; dx \;=\; 1/3.$$

Also, the expected value of $Y^2$ is

$$\mathbf{E}[Y^2] \;=\; \int_0^1 \int_x^{x+1} y^2\; dy\; dx \;=\; \int_0^1 (3x^2+3x+1)/3\; dx \;=\; 7/6.$$

Finally, the expected value of $XY$ is

$$\begin{aligned}\mathbf{E}[XY] \;&=\; \int_0^1 x \int_x^{x+1} y\; dy\; dx \\ &=\; \int_0^1 (2x^2+x)/2\; dx\; dy \;=\; 7/12.\end{aligned}$$

Substituting these into (1), we get $\text{var}(X+Y) = 1/3 + 7/6 + 7/6 - 9/4 = 5/12$.

***Alternative (shortcut) solution to parts (c) and (d)***

Given any value of $X$ (in ([0,1]), we observe that $Y - X$ takes values between 0 and 1, and is uniformly distributed. Since the conditional distribution of $Y - X$ is the same for every value of $X$ in [0,1], we see that $Y - X$ independent of $X$. Thus: (a) $X$ is uniform, and (b) $Y = X + U$, where $U$ is also uniform and independent of $X$. It follows that $\mathbf{E}[X+Y] = \mathbf{E}[2X+U] = 3/2$. Furthermore, $\text{var}(X+Y) = 4\,\text{var}(X) + \text{var}(U) = 5/12$.

6. (a) Let $A$ be the event that the first coin toss resulted in heads. To calculate the probability $\mathbf{P}(A)$, we use the continuous version of the total probability theorem:

$$\mathbf{P}(A) = \int_0^1 \mathbf{P}(A \mid P = p) f_P(p)\; dp = \int_0^1 p(1+\sin(2\pi p))\; dp,$$

which after some calculation yields

$$\mathbf{P}(A) = \frac{\pi - 1}{2\pi}.$$

(b) Using Bayes rule,

$$\begin{aligned} f_{P|A}(p) \;&=\; \frac{\mathbf{P}(A \mid P = p) f_P(p)}{\mathbf{P}(A)} \\ &=\; \begin{cases} \dfrac{2\pi p(1+\sin(2\pi p))}{\pi - 1}, & \text{if } 0 \le p \le 1, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

(c) Let $B$ be the event that the second toss resulted in heads. We have

$$\begin{aligned}\mathbf{P}(B \mid A) &= \int_0^1 \mathbf{P}(B \mid P = p, A) f_{P|A}(p)\ dp \\ &= \int_0^1 \mathbf{P}(B \mid P = p) f_{P|A}(p)\ dp \\ &= \frac{2\pi}{\pi - 1} \int_0^1 p^2 (1 + \sin(2\pi p))\ dp.\end{aligned}$$

After some calculation, this yields

$$\mathbf{P}(B \mid A) = \frac{2\pi}{\pi - 1} \cdot \frac{2\pi - 3}{6\pi} = \frac{2\pi - 3}{3\pi - 3} \approx 0.5110.$$

G1†. Let $a = (\cos\theta, \sin\theta)$ and $b = (b_x, b_y)$. We will show that no point of $R$ lies outside $C$ if and only if

$$|b| \leq |\sin\theta|, \qquad \text{and} \qquad |a| \leq |\cos\theta| \tag{2}$$

The other two vertices of $R$ are $(\cos\theta, b_y)$ and $(b_x, \sin\theta)$. If $|b_x| \leq |\cos\theta|$ and $|b_y| \leq |\sin\theta|$, then each vertex $(x, y)$ of $R$ satisfies $x^2 + y^2 \leq \cos^2\theta + \sin^2\theta = 1$ and no points of $R$ can lie outside of $C$. Conversely if no points of $R$ lie outside $C$, then applying this to the two vertices other than $a$ and $b$, we find

$$\cos^2\theta + b^2 \leq 1, \qquad \text{and} \qquad a^2 + \sin^2\theta \leq 1.$$

which is equivalent to 2.

These conditions imply that $(b_x, b_y)$ lies inside or on $C$, so for any given $\theta$, the probability that the random point $b = (b_x, b_y)$ satisfies (2) is

$$\frac{2|\cos\theta| \cdot 2|\sin\theta|}{\pi} = \frac{2}{\pi}|\sin(2\theta)|$$

and the overall probability is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{2}{\pi}|\sin(2\theta)| d\theta = \frac{4}{\pi^2}\int_0^{\pi/2} \sin(2\theta) d\theta = \frac{4}{\pi^2}$$

.