Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Problem Set 5
Due October 18, 2010

1. Random variables X and Y are distributed according to the joint PDF

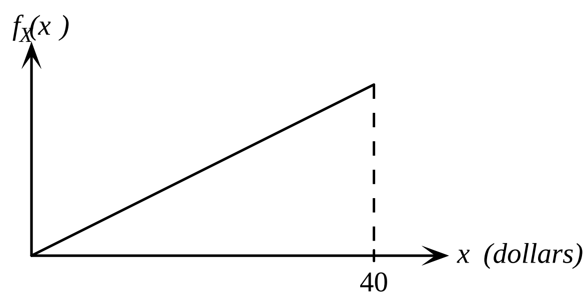
$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.

   (b) Determine the marginal PDF $f_Y(y)$.

   (c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.

2. Paul is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with the PDF shown in the figure. At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in.

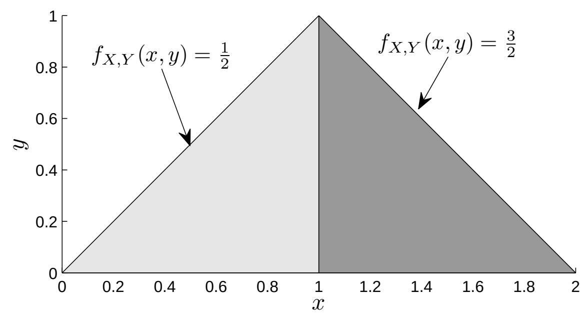


   (a) Determine the joint PDF $f_{X,Y}(x,y)$. Be sure to indicate what the sample space is.

   (b) What is the probability that on any given night Paul makes a positive profit at the casino? Justify your reasoning.

   (c) Find and sketch the probability density function of Paul's profit on any particular night, $Z = Y - X$. What is $\mathbf{E}[Z]$? Please label all axes on your sketch.

3. X and Y are continuous random variables. X takes on values between 0 and 2 while Y takes on values between 0 and 1. Their joint pdf is indicated below.



   (a) Are X and Y independent? Present a convincing argument for your answer.
   (b) Prepare neat, fully labelled plots for $f_X(x)$, $f_{Y|X}(y \mid 0.5)$, and $f_{X|Y}(x \mid 0.5)$.
   (c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.
   (d) Let $W = Y - X$ and determine the cumulative distribution function (CDF) of W.

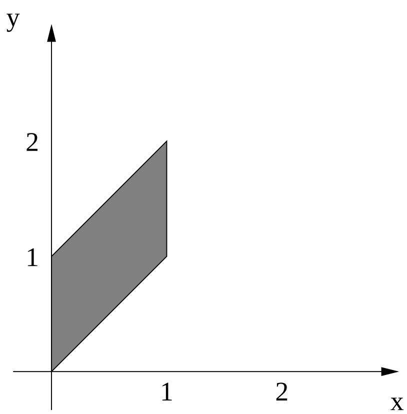
4. **Signal Classification:** Consider the communication of binary-valued messages over some transmission medium. Specifically, any message transmitted between locations is one of two possible symbols, 0 or 1. Each symbol occurs with equal probability. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additive noise that is independent of X. The noise N is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$.

   (a) Suppose the transmitter encodes the symbol 0 with the value $X = -2$ and the symbol 1 with the value $X = 2$. At the other end, the received message is decoded according to the following rules:
      - If $Y \geq 0$, then conclude the symbol 1 was sent.
      - If $Y < 0$. then conclude the symbol 0 was sent.

      Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.
   (b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the symbols with a repeated scheme. The symbol 0 is encoded with the vector $\overline{X} = [-2, -2, -2]^\intercal$ and the symbol 1 is encoded with the vector $\overline{X} = [2, 2, 2]^\intercal$. The vector $\overline{Y} = [Y_1, Y_2, Y_3]^\intercal$ received at the other end is described by $\overline{Y} = \overline{X} + \overline{N}$. The vector $\overline{N} = [N_1, N_2, N_3]^\intercal$ represents the noise vector where each N_i is a random variable assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. Assume each N_i is independent of each other and independent of the X_i's. Each component value of $\overline{Y}$ is decoded with the same rule as in part (a). The receiver then uses a majority rule to determine which symbol was sent. The receiver's decoding rules are:
      - If 2 or more components of $\overline{Y}$ are greater than 0, then conclude the symbol 1 was sent.
      - If 2 or more components of $\overline{Y}$ are less than 0, then conclude the symbol 0 was sent.

      Determine the probability of error for this modified encoding/decoding scheme. Reduce your calculations to a single numerical value.

5. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,1)$.



   (a) Are X and Y independent?
   (b) Find the marginal PDFs of X and Y.
   (c) Find the expected value of $X+Y$.
   (d) Find the variance of $X+Y$.

6. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} 1+\sin(2\pi p), & \text{if } p \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

   In essence, a specific coin produced by this machine will have a fixed probability $P = p$ of giving heads, but you do not know initially what that probability is. A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

   (a) Find the probability that the first coin toss results in heads.
   (b) Given that the first coin toss resulted in heads, find the conditional PDF of P.
   (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the second toss.

G1†. Let C be the circle $\{(x,y) \mid x^2+y^2 \leq 1\}$. A point a is chosen randomly on the boundary of C and another point b is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal ab. What is the probability that no point of R lies outside of C?

†Required for 6.431; optional for 6.041