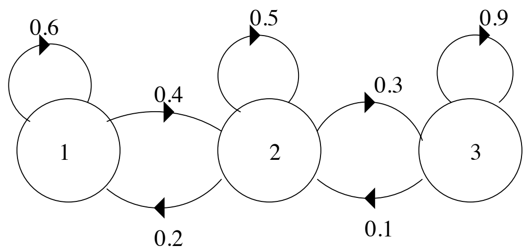
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Final Exam | Fall 2010)

Problem 1. (32 points) Consider a Markov chain $\{X_n; n = 0, 1, \ldots\}$, specified by the following transition diagram.



1. **(4 points)** Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

2. **(4 points)** Find the steady-state probabilities π_1, π_2, π_3 of the different states.

   In case you did not do part (b) correctly, in **all** *subsequent parts of this problem you can just use the symbols π_i: you do not need to plug in actual numbers.*

3. **(4 points)** Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the nth transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbf{P}(Y_n = 1)$.

4. **(4 points)** Is the sequence Y_n a Markov chain? Justify your answer.

5. **(4 points)** Given that the nth transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

6. **(4 points)** Suppose that $X_0 = 1$. Let T be defined as the first *positive time* at which the state is again equal to 1. Show how to find $\mathbf{E}[T]$. (It is enough to write down whatever equation(s) needs to be solved; you do not have to actually solve it/them or to produce a numerical answer.)

7. **(4 points)** Does the sequence $X_1, X_2, X_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

8. **(4 points)** Let $Z_n = \max\{X_1, \ldots, X_n\}$. Does the sequence $Z_1, Z_2, Z_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

Problem 2. (68 points) Alice shows up at an Athena* cluster at time zero and spends her time exclusively in typing emails. The times that her emails are sent are a Poisson process with rate λ_A per hour.

1. **(3 points)** What is the probability that Alice sent exactly three emails during the time interval $[1, 2]$?

2. Let Y_1 and Y_2 be the times at which Alice's first and second emails were sent.

   (a) **(3 points)** Find $\mathbf{E}[Y_2 \mid Y_1]$.

   (b) **(3 points)** Find the PDF of Y_1^2.

   (c) **(3 points)** Find the joint PDF of Y_1 and Y_2.

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*Athena is MIT's UNIX-based computing environment. OCW does not provide access to it.

3. You show up at time 1 and you are told that Alice has sent exactly one email so far. (Only give answers here, no need to justify them.)

   (a) **(3 points)** What is the conditional expectation of Y_2 given this information?

   (b) **(3 points)** What is the conditional expectation of Y_1 given this information?

4. Bob just finished exercising (without email access) and sits next to Alice at time 1. He starts typing emails at time 1, and fires them according to an independent Poisson process with rate λ_B.

   (a) **(5 points)** What is the PMF of the total number of emails sent by the two of them together during the interval $[0, 2]$?

   (b) **(5 points)** What is the expected value of the total typing time associated with the email that Alice is typing at the time that Bob shows up? (Here, "total typing time" includes the time that Alice spent on that email both before and after Bob's arrival.)

   (c) **(5 points)** What is the expected value of the time until each one of them has sent at least one email? (Note that we count time starting from time 0, and we take into account any emails possibly sent out by Alice during the interval $[0, 1]$.)

   (d) **(5 points)** Given that a total of 10 emails were sent during the interval $[0, 2]$, what is the probability that exactly 4 of them were sent by Alice?

5. **(5 points)** Suppose that $\lambda_A = 4$. Use Chebyshev's inequality to find an upper bound on the probability that Alice sent at least 5 emails during the time interval $[0, 1]$. Does the Markov inequality provide a better bound?

6. **(5 points)** You do not know λ_A but you watch Alice for an hour and see that she sent exactly 5 emails. Derive the maximum likelihood estimate of λ_A based on this information.

7. **(5 points)** We have reasons to believe that λ_A is a large number. Let N be the number of emails sent during the interval $[0, 1]$. Justify why the CLT can be applied to N, and give a precise statement of the CLT in this case.

8. **(5 points)** Under the same assumption as in last part, that λ_A is large, you can now pretend that N is a normal random variable. Suppose that you observe the value of N. Give an (approximately) 95% confidence interval for λ_A. State precisely what approximations you are making.
   Possibly useful facts: The cumulative normal distribution satisfies $\Phi(1.645) = 0.95$ and $\Phi(1.96) = 0.975$.

9. You are now told that λ_A is actually the realized value of an exponential random variable Λ, with parameter 2:
$$f_\Lambda(\lambda) = 2e^{-2\lambda}, \qquad \lambda \geq 0.$$

   (a) **(5 points)** Find $\mathbf{E}[N^2]$.

   (b) **(5 points)** Find the linear least squares estimator of Λ given N.

Massachusetts Institute of Technology

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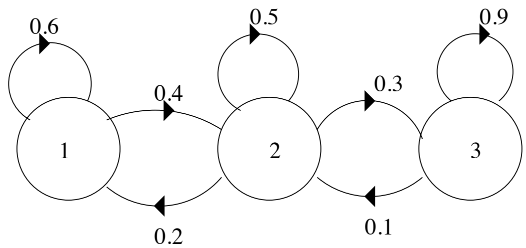

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