Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Solutions to Quiz 2: Spring 2006

Problem 1: (30 points)
Each of the following statements is either True or False. There will be **no partial credit** given for the True False questions, thus any explanations will not be graded. Please **clearly** indicate True or False in the below, ambiguous marks will receive zero credit. All parts have equal weight.

Points breakdown: Each question carries 3 points

(a) X and Y are independent random variables. X is uniformly distributed on the interval $[-2, 2]$, while Y is uniformly distributed on the interval $[-1, 5]$. If $Z = X + Y$, then $f_Z(3) = 1/6$.

True

Since $Z = X + Y$ and X,Y are independent, the PDF of Z ($f_Z(z)$) can be obtained by convolving the PDFs of X ($f_X(x)$) and Y ($f_Y(y)$). That is,

$$f_Z(z) = \int f_X(u) f_Y(z-u)\, du = \int f_Y(u) f_X(z-u)\, du.$$

However, since we are interested in only $f_Z(3)$, we need to evaluate the convolution integral at only one point ($z = 3$). The convolution sum and the corresponding figure are shown below.

$$f_Z(3) = \int_{u=1}^{u=5} f_Y(u) f_X(3-u)\, du = \int_{u=1}^{u=5} (\frac{1}{6})(\frac{1}{4})\, du = \frac{1}{24}4 = \frac{1}{6}$$

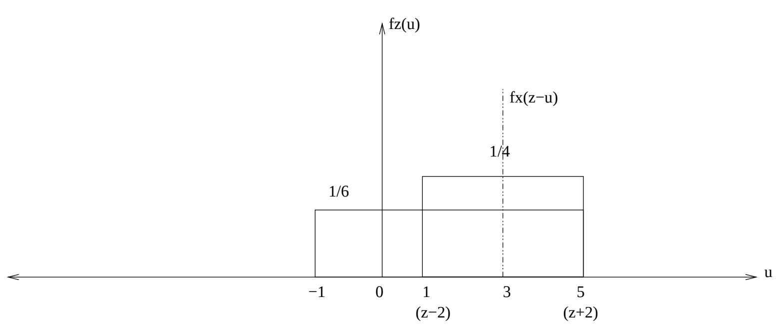

Figure 1: $f_Y(u)$ is uniform between -1 and 5. $f_X(z-u)$ is uniform between $z-2$ and $z+2$.

(b) If X is a Gaussian random variable with zero mean and variance equal to 1, then the density function of $Z = |X|$ is equal to $2f_X(z), z \geq 0$.

True

Here Z is a derived random variable defined by $Z = |X|$. We can obtain the PDF of Z using the standard technique of finding the CDF of Z and then obtaining the PDF by differentiating the CDF. We have (for $z \geq 0$)

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(|X| \leq z) = \mathbf{P}(-z \leq X \leq z) = F_X(z) - F_X(-z).$$

Taking derivatives,

$$\begin{aligned} f_Z(z) &= f_X(z) + f_X(-z) \\ &= 2f_X(z) \text{ since the standard Gaussian PDF is symmetric} \end{aligned}$$

(c) The sum of a random number of independent Gaussian random variables with zero mean and unit variance results in a Gaussian random variable regardless of the distribution of N (the number of sums).

False

This can be verified by taking the transform of the new random variable. If $X_1, X_2 \dots X_N$ are IID Gaussian random variables and N is also a random variable independent of the X_is, then the transform of the sum $Y = X_1 + X_2 \dots X_N$ is given by

$$M_Y(s) = M_N(s)|_{e^s = M_X(s)}.$$

This does not take the form $e^{s\mu} e^{\frac{s^2\sigma^2}{2}}$ for all $M_N(s)$.

(d) If X and Y are independent random variables, both exponentially distributed with parameters λ_1 and λ_2 respectively. Then the random variable $Z = \min\{X, Y\}$ is also exponentially distributed.

True

Here Z is a derived random variable defined as $Z = \min\{X, Y\}$. We can obtain the PDF of Z by first determining its CDF and then taking the derivative. The CDF of Z is given by

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(\min\{X, Y\} \le z).$$

It is not very straight forward to determine this probability. Instead, we can easily obtain $\mathbf{P}(Z \ge z)$. Since this is equivalent to $1 - F_Z(z)$, we have

$$\begin{aligned} 1 - F_Z(z) &= \mathbf{P}(Z > z) \\ &= \mathbf{P}(\min\{X, Y\} > z) \\ &= \mathbf{P}(X > z, Y > z) \\ &= \mathbf{P}(X > z) P(Y > z) \text{ since X,Y are independent} \\ &= \exp(-\lambda_1 z)\exp(-\lambda_2 z) \\ &= \exp(-(\lambda_1 + \lambda_2) z) \end{aligned}$$

Taking the derivative, we have

$$f_Z(z) = (\lambda_1 + \lambda_2)\exp(-(\lambda_1 + \lambda_2) z).$$

This is the pdf of an exponential random variable with parameter $(\lambda_1 + \lambda_2)$.

(e) Let the transform associated with a random variable X be

$$M_X(s) = \left(\frac{e^s}{1-s}\right)^{15}.$$

Then $\mathbf{E}[X]$ is equal to 30.

True

A straight forward way to confirm this fact is to compute the expected value by taking the derivative of $M_X(s)$ and then evaluating it at $s = 0$. We have

$$\begin{aligned}
\frac{dM_X(s)}{ds} &= 15\left(\frac{e^s}{1-s}\right)^{15}\left(\frac{(1-s)e^s - e^s(-1)}{(1-s)^2}\right) \\
\mathbf{E}[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0} &= 15 \cdot 2 = 30
\end{aligned}$$

The next set of questions are concerned with two independent random variables: Y is normal with mean 0 and variance 1, and X is uniform between $[0, 1]$. $Z = X + Y$.

(f) The conditional density of Z given X, $f_{Z|X}(z|x)$, is normal with mean x and variance 1.

True

Given the value of x, the random variable Z is a derived random variable given by $Z = x+Y$. This is a normal random variable with mean $x + \mathbf{E}[Y]$ and variance $\text{var}(Y)$.

(g) $\text{var}(Z) = 2$.

False

$$\begin{aligned}
\text{var}(Z) &= \text{var}(X) + \text{var}(Y) \quad \text{since } X \text{ and } Y \text{ are independent} \\
&= \frac{1}{12} + 1 \neq 2
\end{aligned}$$

(h) $\mathbf{E}[X \mid Z = -1] = -1$.

False Since X is uniformly distributed between 0 and 1, the expected value cannot take on negative values.

(i) $\text{cov}(X, Z) = \text{var}(X)$

True By definition, we have $\text{cov}(X, Z) = \mathbf{E}[XZ] - \mathbf{E}[X]\mathbf{E}[Z]$. Using $Z = X + Y$, we have

$$\begin{aligned}
\text{cov}(X, Z) &= \mathbf{E}[X(X+Y)] - \mathbf{E}[X]\mathbf{E}[X+Y] \\
&= \mathbf{E}[X^2] + \mathbf{E}[XY] - \mathbf{E}[X](\mathbf{E}[X] + \mathbf{E}[Y]) \\
&= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 + \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \text{var}(X) + \text{cov}(X, Y)
\end{aligned}$$

Since X and Y are independent, $\text{cov}(X, Y) = 0$.

(j) $Z = \mathbf{E}[X \mid Z] + \mathbf{E}[Y \mid Z]$

True

Since $Z = X + Y$, we have

$$\begin{aligned}
\mathbf{E}[Z|Z] &= \mathbf{E}[X+Y|Z] \quad \text{conditional expectation is linear} \\
Z &= \mathbf{E}[X|Z] + \mathbf{E}[Y|Z] \quad \text{since } \mathbf{E}[Z|Z] = Z
\end{aligned}$$

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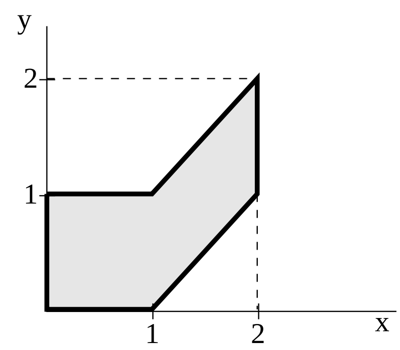
Problem 2: (25 points)

Points breakdown: (a) 7 points; (b)–(d) 6 points each

The continuous random variables X and Y have a joint pdf given by



$$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } (x,y) \text{ belongs to the shaded region;} \\ 0, & \text{otherwise.} \end{cases}$$

In class we have shown the minimum least squares estimate of Y is given by $\mathbf{E}[Y \mid X = x]$

(a) Find the least squares estimate of Y given that $X = x$, for all possible values of x. For full credit write the functional form, as opposed to a graph.

The least square estimate of Y based on X is given by $\mathbf{E}[Y \mid X]$. In order to determine this quantity, we need to evaluate the conditional density $f_{Y|X}(y|x)$, for all values of x and y. Since the joint density is uniform through out the specified region, the conditional density will also be uniform and is given by

$$f_{Y|X}(y|x) = \begin{cases} 1, & 0 \le y \le 1, & x \le 1 \\ 1, & x-1 \le y \le x, & 1 < x \le 2 \end{cases} \quad (1)$$

The conditional expectations follow naturally from this.

$$\boxed{\mathbf{E}[Y|X] = \begin{cases} \frac{1}{2} & 0 \le X \le 1 \\ X - \frac{1}{2} & 1 < X \le 2 \end{cases}}$$

(b) Let $g(x)$ be the estimate from part (a). Find $\mathbf{E}[g(X)]$ and $\mathrm{var}(g(X))$.

$g(X)$ is a derived random variable that is defined as

$$g(X) = \begin{cases} \frac{1}{2}, & 0 \le X \le 1 \\ X - \frac{1}{2}, & 1 < X \le 2 \end{cases}$$

The expected value of $g(X)$ is given by $\mathbf{E}[g(X)] = \int g(x) f_X(x)\, dx$. The marginal density of X ($f_X(x)$) can be obtained by integrating the joint density. (It is easy to show that $c = 0.5$, since the total volume $2c$ should be unity).

$$f_X(x) = \begin{cases} \int_{y=0}^{y=1} c\,dy & = \frac{1}{2}, \quad 0 < x \leq 1 \\ \int_{y=x-1}^{y=x} c\,dy & = \frac{1}{2}, \quad 1 < x \leq 2 \end{cases}$$

Thus we have

$$\mathbf{E}[g(X)] = \int_{x=0}^{1} (\frac{1}{2})(\frac{1}{2})\,dx + \int_{x=1}^{2} (\frac{1}{2})(x - \frac{1}{2})\,dx = \frac{3}{4}$$

To compute the variance of $g(X)$, we compute $\mathbf{E}[g(X)^2]$ which is given by

$$\mathbf{E}[g(X)^2] = \int_{x=0}^{1} (\frac{1}{2})(\frac{1}{2})^2\,dx + \int_{x=1}^{2} (\frac{1}{2})(x - \frac{1}{2})^2\,dx = \frac{2}{3}$$

The variance $\text{var}(g(X))$ is obtained by using

$$\text{var}(g(X)) = \mathbf{E}[g(X)^2] - (\mathbf{E}[g(X)])^2 = \frac{2}{3} - \left(\frac{3}{4}\right)^2 = \frac{5}{48} \approx 0.104$$

(c) Find the mean square error $\mathbf{E}[(Y - g(X))^2]$. Is it the same as $\mathbf{E}[\text{var}(Y|X)]$?

The mean square error $\mathbf{E}[(Y - g(X))^2]$ is a function of both Y and X. In general, we have to evaluate this quantity by evaluating the mean of $h(X,Y) = (Y - g(X))^2$ over the joint density $f_{X,Y}(x,y)$. However, we can simplify this by using iterated expectation.

$$\begin{aligned} \mathbf{E}[(Y - g(X))^2] &= \mathbf{E}[\mathbf{E}[(Y - g(X))^2 \mid X]] \\ &= \mathbf{E}[\mathbf{E}[(Y - \mathbf{E}[Y|X])^2 \mid X]] \quad \text{since } g(X) = \mathbf{E}[Y|X] \\ &= \mathbf{E}[\text{var}(Y|X)] \quad \text{since } \mathbf{E}[(Y - \mathbf{E}[Y|X])^2 \mid X] = \text{var}(Y|X) \end{aligned}$$

Since $f_{Y|X}$ is uniform for all values of X as seen before, we have $\text{var}(Y|X) = \frac{1}{12}, 0 < X < 2$. Furthermore, we know that $f_X(x)$ is also uniform in this interval. Thus,

$$\mathbf{E}[\text{var}(Y|X)] = \int_{x=0}^{2} \frac{1}{2}\frac{1}{12}\,dx = \frac{1}{12}$$

(d) Find $\text{var}(Y)$.

Using total variance theorem, we have

$$\begin{aligned} \text{var}(Y) &= \mathbf{E}[\text{var}(Y|X)] + \text{var}[\mathbf{E}[Y|X]] \\ &= \mathbf{E}[\frac{1}{12}] + \text{var}[g(X)] \text{ from part (b)} \\ &= \frac{1}{12} + \frac{5}{48} = \frac{3}{16} = 0.1875 \end{aligned}$$

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Problem 3: (42 points)
Points breakdown: (a-g) 6 points each

Please write all work for Problem 3 in your **second blue book**. No work recorded below will be graded. All parts have approximately the same weight.

Each year, a publisher sends Professor MD a random number of text books to review. The number of books Professor MD receives each year can be modeled as a Poisson random variable N, with mean μ. Each book contains a random number of typos, where the number of typos in one book can be modeled as a Poisson random variable with mean λ. Let B_i denote the number of typos in book i. Assume N is independent of B_i for all i, and B_i is independent of B_j for all $i \neq j$. Professor MD is an expert in the field of typo identification, but even experts aren't perfect. Assume Professor MD finds any existing typo with probability p, independent of finding any other typos as well as N and B_i.

The publisher offers Professor MD two different annual salary options for reviewing the text books. The two options are:

Option 1: 1 dollar for each typo found.

Option 2: 1 dollar for each book where at least one typo is found.

Let X_i be the amount of money Professor MD receives for book i, and let T be be the total amount of money Professor MD receives in any given year.

(a) Find and correctly state the PMF of X_i under option 1. For full credit reduce this expression to a well known PMF. What's the name of this PMF?

Let Y_i be a Bernoulli random variable,

$$Y_i = \begin{cases} 1 & \text{If the } i\text{th typo is found,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the pdf of Y_i is

$$p_{Y_i}(y) = \begin{cases} p & \text{If } y = 1, \\ 1-p & \text{if } y = 0. \end{cases}$$

Now, note that $X_i = Y_1 + Y_2 + \ldots + Y_{B_i}$. So, $M_{X_i}(s) = M_{B_i}(s)|_{e^s = M_Y(s)}$, where $M_Y(s) = 1 - p + pe^s$ and $M_{B_i}(s) = e^{\lambda(e^s-1)}$. And, substituting yields:

$$M_{X_i}(s) = M_{B_i}(s)|_{e^s=M_Y(s)} = e^{\lambda(1-p+pe^s-1)} = e^{\lambda p(e^s-1)}.$$

Thus, $\boxed{X_i \text{ is Poisson with parameter } \lambda p}$ and its pmf is

$$p_{X_i}(x) = \frac{(\lambda p)^x e^{-\lambda p}}{x!} \qquad x = 0, 1, 2, \ldots$$

(b) Find $M_T(s)$ under option 1.

(b) $T = X_1+X_2+\ldots+X_N$. Again, we use $M_T(s) = M_N(s)|_{e^s=M_X(s)}$, where $M_X(s) = e^{\lambda p(e^s-1)}$ and $M_N(s) = e^{\mu(e^s-1)}$. So, $\boxed{M_T(s) = e^{\mu(e^{\lambda p(e^s-1)}-1)}.}$

(c) Find $\mathbf{P}(T = 2)$ under option 1.

Because T is a discrete R.V. that takes nonegative integer values, $\mathbf{P}(T = 2) = \frac{1}{2!}\frac{d^2}{d(e^s)^2}M_T(s)|_{e^s=0}$. We have,

$$\begin{aligned} \frac{d}{d(e^s)}M_T(s) &= \mu\lambda p e^{\lambda p(e^s-1)}e^{\mu(e^{\lambda p(e^s-1)}-1)}, \\ \frac{d^2}{d(e^s)^2}M_T(s) &= \mu(\lambda p)^2 e^{\lambda p(e^s-1)}e^{\mu(e^{\lambda p(e^s-1)}-1)} + \left(\mu\lambda p e^{\lambda p(e^s-1)}\right)^2 e^{\mu(e^{\lambda p(e^s-1)}-1)}. \end{aligned}$$

So, $\boxed{\mathbf{P}(T = 2) = \frac{1}{2}\mu(\lambda p)^2 e^{-\lambda p}e^{\mu(e^{-\lambda p}-1)}(1 + \mu e^{-\lambda p}).}$

(d) Find $\mathbf{E}[T]$ under option 1.

$\mathbf{E}[T] = \mathbf{E}[B_i]\mathbf{E}[Y]\mathbf{E}[N] = \boxed{\mu\lambda p.}$

(e) Find var(T) under option 1.

$\text{Var}(T) = \text{Var}(X_i)\mathbf{E}[N] + \text{Var}(N)(\mathbf{E}[X_i])^2$, where $\mathbf{E}[X_i] = \lambda p$, $\text{Var}(X_i) = \lambda p$, $\mathbf{E}[N] = \mu$, and $\text{Var}(N) = \mu$. So, $\boxed{\text{Var}(T) = \mu\lambda p(1 + \lambda p).}$

(f) Find and correctly state the PMF of X_i under option 2. For full credit reduce this expression to a well known PMF. What's the name of this PMF?

Let X_i be a Bernoulli random variable that is defined as follows,

$$X_i = \begin{cases} 1 & \text{If at least one typo is found in book } i, \\ 0 & \text{if no typos are found.} \end{cases}$$

Note that X_i is also the amount of money MD receives for book i under option 2. Let Z_i be the number of typos found in book B_i. From part a, we know that Z_i is a possion random variable with parameter λp. Now, $P(X_i = 1) = P(Z_i > 0)$ and, $\mathbf{P}(X_i = 0) = P(Z_i = 0) = e^{-\lambda p}$,. So, the pmf of X_i is,

$$p_{X_i}(x_i) = \begin{cases} 1 - e^{-\lambda p} & \text{If } x_i = 1, \\ e^{-\lambda p} & \text{if } x_i = 0. \end{cases}$$

It is to be noted that $\boxed{X_i \text{ is a Bernoulli random variable with parameter } 1 - e^{-\lambda p}}$.

(g) Find $\mathbf{E}[T]$ under option 2. Hint: Fully reduce your answer in (f) before attempting.

Under option 2, $T = X_1 + X_2 + \ldots + X_N$. So, $\mathbf{E}[T] = \mathbf{E}[X_i]\mathbf{E}[N] = \boxed{\mu(1 - e^{-\lambda p}).}$