Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 2: Solutions
September 14, 2010

1. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B | A)$ and $\mathbf{P}(A \cap B | A \cup B)$. We have

$$\mathbf{P}(A \cap B | A) = \frac{\mathbf{P}((A \cap B) \cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)},$$

and

$$\mathbf{P}(A \cap B | A \cup B) = \frac{\mathbf{P}((A \cap B) \cap (A \cup B))}{\mathbf{P}(A \cup B)} = \frac{A \cap B}{A \cup B}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH, HT, TH, TT are equally likey, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \qquad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = 1/3.$$

A generalization of Alice's reasoning is that if A, B, and C are events such that $B \subset C$ and $A \cap B = A \cap C$ (for example, if $A \subset B \subset C$), then the event A is at least as likely if we know that B has occurred than if we know that C has occurred. Alice's reasoning corresponds to the special case where $C = A \cup B$.

2. (a) Each possible outcome has probability $1/36$. There are 6 possible outcomes that are doubles, so the probability of doubles is $6/36 = 1/6$.

   (b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\},$$

   2 of which are doubles, so the conditional probability of doubles is $2/6 = 1/3$.

   (c) There are 11 possible outcomes with at least one 6, namely, $(6,6)$, $(6,i)$, and $(i,6)$, for $i = 1, 2, \ldots, 5$. Thus, the probability that at least one die is a 6 is $11/36$.

   (d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $10/30 = 1/3$.

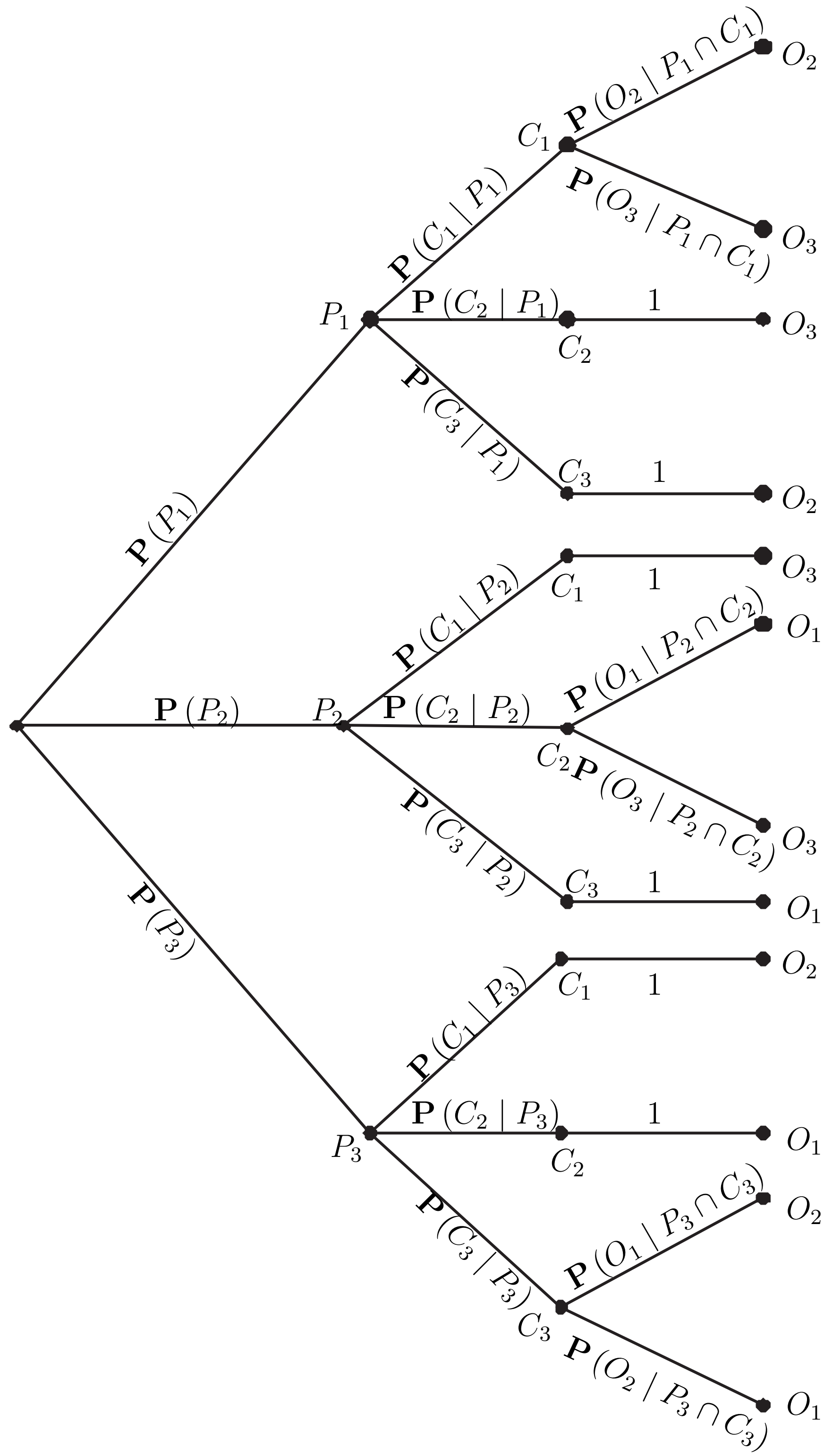
3. (a) See the textbook, Example 1.13, page 29.

   (b) See the textbook, Example 1.17, page 33.

4. See the textbook, Example 1.12 (The Monty Hall Problem), page 27.

   An alternative solution is given below:

   Let P_i denote the event where the prize is behind door i, C_i denote the event where you initially choose door i, and O_i denote the event where your friend opens door i. The corresponding probability tree is:


P(P1)
P(P2)
P(P3)
P1
P2
P3
P(C1 | P1)
P(C2 | P1)
P(C3 | P1)
P(C1 | P2)
P(C2 | P2)
P(C3 | P2)
P(C1 | P3)
P(C2 | P3)
P(C3 | P3)
C1
C2
C3
P(O2 | P1 ∩ C1)
P(O3 | P1 ∩ C1)
P(O1 | P2 ∩ C2)
P(O3 | P2 ∩ C2)
P(O1 | P3 ∩ C3)
P(O2 | P3 ∩ C3)
1
O1
O2
O3

(a) The probability of winning when not switching from your initial choice is the probability that the prize is behind the door you initially chose:

$$\begin{aligned}
\mathbf{P}(\text{Win when not switching}) &= \mathbf{P}(P_1 \cap C_1) + \mathbf{P}(P_2 \cap C_2) + \mathbf{P}(P_3 \cap C_3) \\
&= \mathbf{P}(P_1)\mathbf{P}(C_1|P_1) + \mathbf{P}(P_2)\mathbf{P}(C_2|P_2) + \mathbf{P}(P_3)\mathbf{P}(C_3|P_3) \\
&= \mathbf{P}(P_1)\mathbf{P}(C_1) + \mathbf{P}(P_2)\mathbf{P}(C_2) + \mathbf{P}(P_3)\mathbf{P}(C_3) \\
&= 1/3 \cdot (\mathbf{P}(C_1) + \mathbf{P}(C_2) + \mathbf{P}(C_3)) \\
&= 1/3
\end{aligned}$$

(b) The probability of winning when switching from your initial choice is the probability that the prize is behind the remaining (unopened) door:

$$\begin{aligned}
\mathbf{P}(\text{Win when switching}) &= \mathbf{P}(P_1 \cap C_2 \cap O_3) + \mathbf{P}(P_1 \cap C_3 \cap O_2) + \mathbf{P}(P_2 \cap C_1 \cap O_3) \\
&\quad + \mathbf{P}(P_2 \cap C_3 \cap O_1) + \mathbf{P}(P_3 \cap C_1 \cap O_2) + \mathbf{P}(P_3 \cap C_2 \cap O_1) \\
&= \mathbf{P}(P_1 \cap C_2) + \mathbf{P}(P_1 \cap C_3) + \mathbf{P}(P_2 \cap C_1) + \mathbf{P}(P_2 \cap C_3) \\
&\quad + \mathbf{P}(P_3 \cap C_1) + \mathbf{P}(P_3 \cap C_2) \\
&= \mathbf{P}(P_1)\mathbf{P}(C_2) + \mathbf{P}(P_1)\mathbf{P}(C_3) + \mathbf{P}(P_2)\mathbf{P}(C_1) + \mathbf{P}(P_2)\mathbf{P}(C_3) \\
&\quad + \mathbf{P}(P_3)\mathbf{P}(C_1) + \mathbf{P}(P_3)\mathbf{P}(C_2) \\
&= 2/3 \cdot (\mathbf{P}(C_1) + \mathbf{P}(C_2) + \mathbf{P}(C_3)) \\
&= 2/3
\end{aligned}$$

(c) Given C_1, that you first choose door 1, with the new strategy of switching only if door 3 is opened, you win if the prize behind door 1 and door 2 is opened or if the prize is behind door 2 and door 3 is opened.

$$\begin{aligned}
\mathbf{P}(\text{Win with new strategy}|C_1) &= \mathbf{P}(P_1 \cap O_2|C_1) + \mathbf{P}(P_2 \cap O_3|C_1) \\
&= \mathbf{P}(P_1|C_1)\mathbf{P}(O_2|P_1 \cap C_1) + \mathbf{P}(P_2|C_1)\mathbf{P}(O_3|P_2 \cap C_1) \\
&= \mathbf{P}(P_1)\mathbf{P}(O_2|P_1 \cap C_1) + \mathbf{P}(P_2)\mathbf{P}(O_3|P_2 \cap C_1) \\
&= 1/3 \cdot \mathbf{P}(O_2|P_1 \cap C_1) + 1/3 \cdot 1 \\
&= 1/3 \cdot (\mathbf{P}(O_2|P_1 \cap C_1) + 1)
\end{aligned}$$

Given that your initial choice is door 1, the probability of winning under this new strategy is dependent on how your friend decides which of doors 2 or 3 to open if the prize also lies behind door 1. If he always picks door 2, then $\mathbf{P}(O_2|P_1 \cap C_1) = 1$ and $\mathbf{P}(\text{Win with new strategy}|C_1) = 2/3$. If he picks between doors 2 and 3 with equal probability then $\mathbf{P}(O_2|P_1 \cap C_1) = 1/2$ and $\mathbf{P}(\text{Win with new strategy}|C_1) = 1/2$.

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