6.041/6.431 Spring 2008 Quiz 2
Wednesday, April 16, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

6.041/6.431: ____________________

Question	Part	Score	Out of
0			3
1	all		36
2	a		4
	b		5
	c		5
	d		8
	e		5
	f		6
3	a		4
	b		6
	c		6
	d		6
	e		6
Total			100

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed 2 two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- We will send out an email with more information on how to obtain your quiz before drop date.
- **Good Luck!**

Question 1: Multiple choice questions. **CLEARLY** circle the best answer for each question below. Each question is worth 4 points each, with no partial credit given.

a. (4 pts) Let X_1, X_2, and X_3 be independent random variables with the continuous uniform distribution over $[0, 1]$. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

   (i) 1/6
   (ii) 1/3
   (iii) 1/2
   (iv) 1/4

b. (4 pts) Let X and Y be two continuous random variables. Then

   (i) $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
   (ii) $\mathbf{E}[X^2 + Y^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2]$
   (iii) $f_{X+Y}(x + y) = f_X(x)f_Y(y)$
   (iv) $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

c. (4 pts) Suppose X is uniformly distributed over $[0, 4]$ and Y is uniformly distributed over $[0, 1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

   (i) $f_Z(4.5) = 0$
   (ii) $f_Z(4.5) = 1/8$
   (iii) $f_Z(4.5) = 1/4$
   (iv) $f_Z(4.5) = 1/2$

d. (4 pts) For the random variables defined in part (c), $\mathbf{P}(\max(X, Y) > 3)$ is equal to

   (i) 0
   (ii) 9/4
   (iii) 3/4
   (iv) 1/4

e. (4 pts) Consider the following variant of the hat problem from lecture: N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $\mathbf{E}$[X] is equal to

   (i) λ
   (ii) $1/\lambda^2$
   (iii) $1/\lambda$
   (iv) 1

f. (4 pts) Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) W is Poisson with parameter $\lambda_1 + \lambda_2$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

g. (4 pts) Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\mathbf{P}(X = 0) = (0.4)^{50}$

(iv) $\mathbf{P}(X = 50) = 0.6$

h. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = x/8$ for $0 \le x \le 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $1 - \Phi\left(\frac{5}{\sqrt{2}}\right)$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

i. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = 1$, $0 \le x \le 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$, for some integer n. Then $\text{var}(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\frac{1}{3^n} - \frac{1}{4^n}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Question 2: Each Mac book has a lifetime that is exponentially distributed with parameter λ. The lifetime of Mac books are independent of each other. Suppose you have two Mac books, which you begin using at the same time. Define T_1 as the time of the first ~~laptop~~ failure and T_2 as the time of the second ~~laptop~~ failure.

a. (4 pts) Compute $f_{T_1}(t_1)$

b. (5 pts) Let $X = T_2 - T_1$. Compute $f_{X|T_1}(x|t_1)$

c. (5 pts) Is X independent of T_1? Give a mathematical justification for your answer.

d. (8 pts) Compute $f_{T_2}(t_2)$ and $\mathbf{E}[T_2]$

e. (5 pts) Now suppose you have 100 Mac books, and let Y be the time of the first laptop failure. Find the best answer for $\mathbf{P}(Y < 0.01)$

Your friend, Charlie, loves Mac books so much he buys S new Mac books every day! On any given day S is equally likely to be 4 or 8, and all days are independent from each other. Let S_{100} be the number of Mac books Charlie buys over the next 100 days.

f. (6 pts) Find the best approximation for $\mathbf{P}(S_{100} \leq 608)$. Express your final answer in terms of $\Phi(\cdot)$, the CDF of the standard normal.

Question 3: Saif is a well intentioned though slightly indecisive fellow. Every morning he flips a coin to decide where to go. If the coin is heads he drives to the mall, if it comes up tails he volunteers at the local shelter. Saif's coin is not necessarily fair, rather it possesses a probability of heads equal to q. We do not know q, but we do know it is well-modeled by a random variable Q where the density of Q is

$$f_Q(q) = \begin{cases} 2q & \text{for } \ 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume conditioned on Q each coin flip is independent. Note parts a, b, c, and $\{d, e\}$ may be answered independent of each other.

a. (4 pts) What's the probability that Saif goes to the local shelter if he flips the coin once?

In an attempt to promote virtuous behavior, Saif's father offers to pay him \$4 every day he volunteers at the local shelter. Define X as Saif's payout if he flips the coin every morning for the next 30 days.

b. (6 pts) Find $\text{var}(X)$

Let event B be that Saif goes to the local shelter at least once in k days.

c. (6 pts) Find the conditional density of Q given B, $f_{Q|B}(q)$

While shopping at the mall, Saif gets a call from his sister Mais. They agree to meet at the Coco Cabana Court yard at exactly 1:30PM. Unfortunately Mais arrives Z minutes late, where Z is a continuous uniform random variable from zero to 10 minutes. Saif is furious that Mais has kept him waiting, and demands Mais pay him R dollars where $R = \exp(Z + 2)$.

d. (6 pts) Find Saif's expected payout, $\mathbf{E}[R]$

e. (6 pts) Find the density of Saif's payout, $f_R(r)$