6.013, Electromagnetic Fields, Forces, and Motion
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September 22, 2005

Lecture 5: The Electric Potential and the Method of Images

I. Nonuniqueness of Voltage in an MQS System

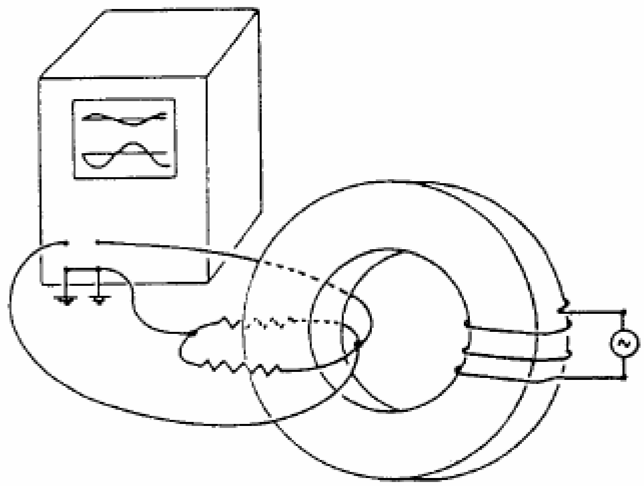


Figure 10.0.1 A pair of unequal resistors are connected in series around a magnetic circuit. Voltages measured between the terminals of the resistors by connecting the nodes to the dual-trace oscilloscope, as shown, differ in magnitude and are 180 degrees out of phase.

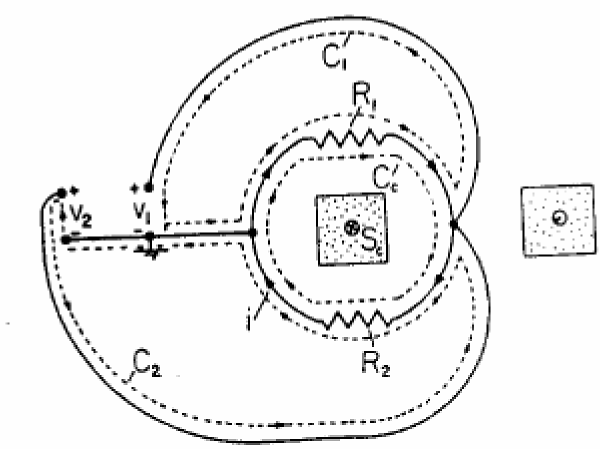


Figure 10.0.2 Schematic of circuit for experiment of Figure 10.0.1, showing contours used with Faraday's law to predict the differing voltages v_1 and v_2.

$$\Phi_\lambda = \int_{S_c} \bar{B} \bullet \overline{da}$$

$$\oint_{C_1} \bar{E} \bullet \overline{ds} = v_1 + iR_1 = 0$$

$$\oint_{C_2} \bar{E} \bullet \overline{ds} = -v_2 + iR_2 = 0$$

$$\oint_{C_c} \bar{E} \bullet \overline{ds} = -\frac{d\Phi_\lambda}{dt} = i\left(R_1 + R_2\right)$$

$$i = -\frac{1}{\left(R_1 + R_2\right)} \frac{d\Phi_\lambda}{dt}$$

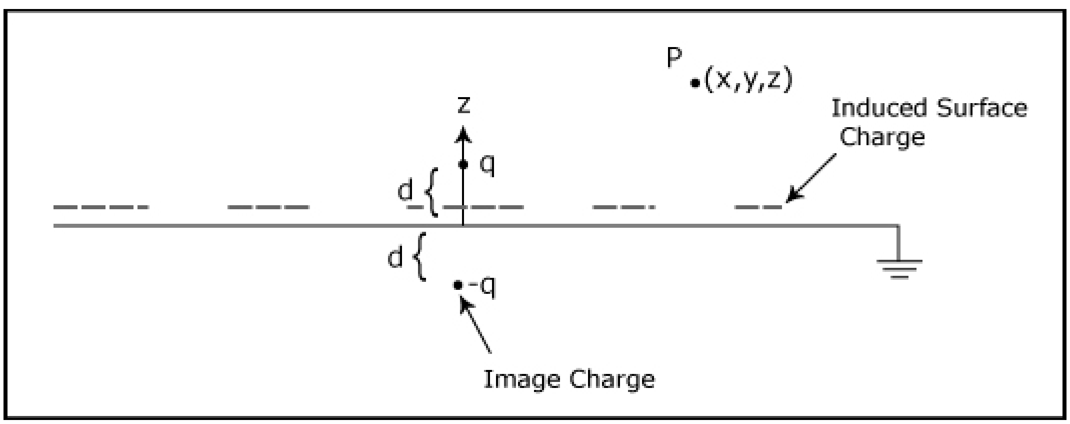
$$v_1 = -iR_1 = \frac{+R_1}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

$$v_2 = iR_2 = \frac{-R_2}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

$$\frac{v_1}{v_2} = -\frac{R_1}{R_2}$$

II. Point Charge Above Ground Plane

1. Potential Electric Field



$$\Phi_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + \left(z - d\right)^2}} - \frac{1}{\sqrt{x^2 + y^2 + \left(z + d\right)^2}}\right]$$

$$\bar{E}_p = -\nabla\Phi_p = -\left[\frac{\partial\Phi_p}{\partial x}\bar{i}_x + \frac{\partial\Phi_p}{\partial y}\bar{i}_y + \frac{\partial\Phi_p}{\partial z}\bar{i}_z\right]$$

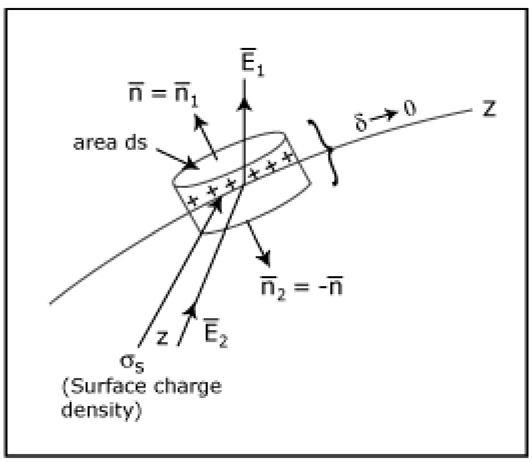
$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\not{2}\left(x\,\bar{i}_x + y\,\bar{i}_y + (z-d)\,\bar{i}_z\right)}{\not{2}\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{\not{2}\left(x\,\bar{i}_x + y\,\bar{i}_y + (z+d)\,\bar{i}_z\right)}{\not{2}\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right]$$

$$\bar{E}_p(z=0) = \frac{q}{2\pi\varepsilon_0}\frac{(-d)}{\left[x^2+y^2+d^2\right]^{3/2}}\bar{i}_z$$

(perpendicular to equipotential ground plane)

2. Gauss's Law Boundary Condition

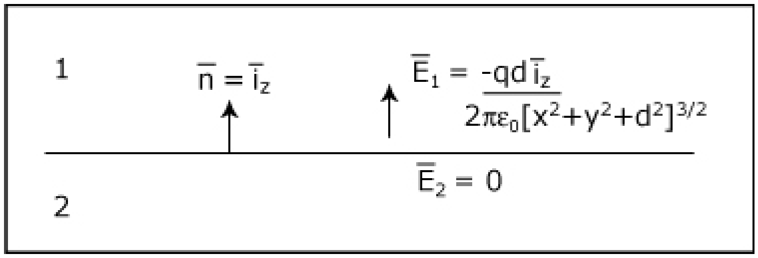
$$\oint_S \varepsilon_0\bar{E}\bullet\overline{da} = \int_V \rho dV$$



$$\oint_S \varepsilon_0\bar{E}\bullet\overline{da} = \left(\varepsilon_0\bar{E}_1\bullet\bar{n}_1 + \varepsilon_0\bar{E}_2\bullet\bar{n}_2\right)dS = \sigma_s dS \quad \text{(total charge inside pillbox)}$$

$$\sigma_s = \varepsilon_0\bar{n}\bullet\left[\bar{E}_1 - \bar{E}_2\right]$$

z=0:



At z=0:

$$\sigma_s = \varepsilon_0 \bar{n} \bullet \left[\bar{E}_1 - \bar{E}_2\right] = \varepsilon_0\, \bar{i}_z \bullet \bar{E}_1 = \varepsilon_0 E_z = \frac{-qd}{2\pi\left[x^2 + y^2 + d^2\right]^{3/2}} = \frac{-qd}{2\pi\left[r^2 + d^2\right]^{3/2}}$$

$$r^2 = x^2 + y^2$$

$$q_T\left(z = 0\right) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} \sigma_s dxdy = \int_{r=0}^{\infty} \int_{\phi=0}^{2\pi} \sigma_s rdrd\phi = \frac{-qd}{\not{2}\not{\pi}}\left(\not{2}\not{\pi}\right) \int_{r=0}^{\infty} \frac{rdr}{\left[r^2 + d^2\right]^{3/2}}$$

$$u = r^2 + d^2 \Rightarrow du = 2rdr$$

$$\int \frac{rdr}{\left[r^2 + d^2\right]^{3/2}} = \int \frac{du}{2u^{3/2}} = -u^{-1/2} = -\frac{1}{\sqrt{r^2 + d^2}}$$

$$q_T\left(z = 0\right) = \left.\frac{+qd}{\sqrt{r^2 + d^2}}\right|_0^{\infty} = -q$$

$$\bar{f}_q = \frac{-q^2}{4\pi\varepsilon_0 \left(2d\right)^2}\, \bar{i}_z = \frac{-q^2}{16\pi\varepsilon_0 d^2}$$

III. Point Charge and Sphere

1. Grounded Sphere

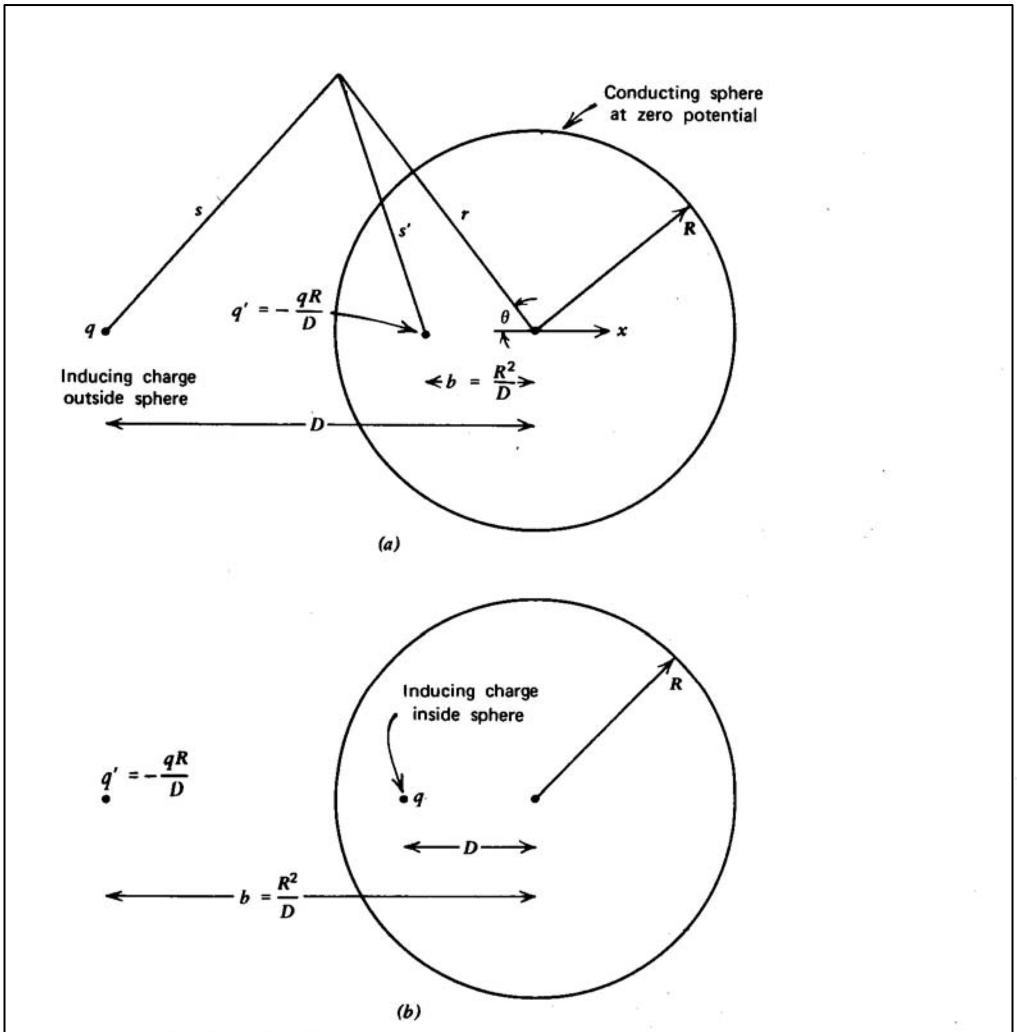


Figure 2-27 (*a*) The field due to a point charge q, a distance D outside a conducting sphere of radius R, can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (*b*) The same relations hold true if the charge q is inside the sphere but now the image charge is outside the sphere, since $D < R$.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s} + \frac{q'}{s'}\right)$$

$$s = \left[r^2 + D^2 - 2rD\cos\theta\right]^{1/2},\ s' = \left[b^2 + r^2 - 2rb\cos\theta\right]^{1/2}$$

$$\Phi(r = R) = 0 \Rightarrow \frac{q}{s} = \frac{-q'}{s'} \Rightarrow \left(\frac{q}{s}\right)^2 = \left(\frac{q'}{s'}\right)^2$$

$$q^2 s'^2 = q'^2 s^2 \Rightarrow q'^2 \left[R^2 + D^2 - 2RD\cos\theta\right] = q^2\left[b^2 + R^2 - 2Rb\cos\theta\right]$$

$$q'^2\left(R^2 + D^2\right) = q^2\left(b^2 + R^2\right)$$

$$+q'^2 \cancel{2}\cancel{R}D\cancel{\cos\theta} = +q^2\cancel{2}\cancel{R}b\cancel{\cos\theta} \Rightarrow \frac{q'^2}{q^2} = \frac{b}{D}$$

$$\frac{b}{D}\left(R^2 + D^2\right) = b^2 + R^2 \Rightarrow b^2 - b\left(\frac{R^2}{D} + D\right) + R^2 = 0$$

$$\left(b - D\right)\left(b - \frac{R^2}{D}\right) = 0$$

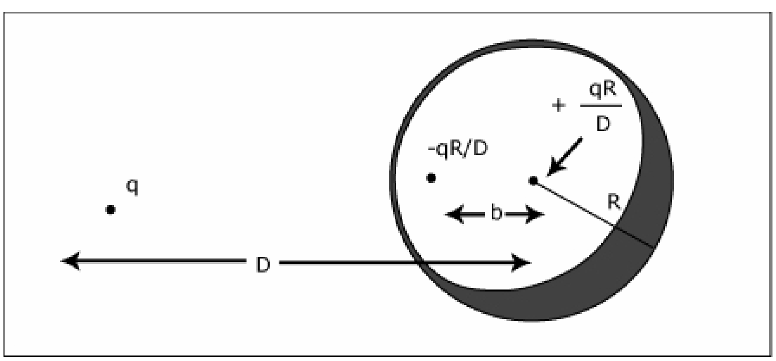
$$b = \frac{R^2}{D}$$

$$q'^2 = q^2\frac{b}{D} = q^2\frac{R^2}{D^2} \Rightarrow q' = -qR/D$$

Force on sphere

$$f_x = \frac{qq'}{4\pi\varepsilon_0\left(D-b\right)^2} = \frac{-q^2R/D}{4\pi\varepsilon_0\left(D - \frac{R^2}{D}\right)^2} = \frac{-q^2RD}{4\pi\varepsilon_0\left(D^2 - R^2\right)^2}$$

2. Isolated Sphere [Put additional Image Charge $+q' = +qR/D$ at center] (zero charge)



$$\Phi\left(r = R\right) = \frac{q'}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0 D}$$

force on sphere

$$f_x = \frac{q}{4\pi\varepsilon_0}\left[\frac{q'}{(D-b)^2} - \frac{q'}{D^2}\right] = \frac{qq'\left[D^2 - (D-b)^2\right]}{4\pi\varepsilon_0 D^2 (D-b)^2} = \frac{-q^2R\left[2bD - b^2\right]}{4\pi\varepsilon_0 D^3\left(D - \frac{R^2}{D}\right)^2}$$

$$f_x = \frac{-q^2RD^2}{4\pi\varepsilon_0 D^3\left(D^2 - R^2\right)^2}\frac{R^2}{D}\left[2D - \frac{R^2}{D}\right] = \frac{-q^2R^3}{4\pi\varepsilon_0 D^3\left(D^2 - R^2\right)^2}\left[2D^2 - R^2\right]$$

IV. Demonstration 4.7.1 – Charge Induced in Ground Plane by Overhead Conductor

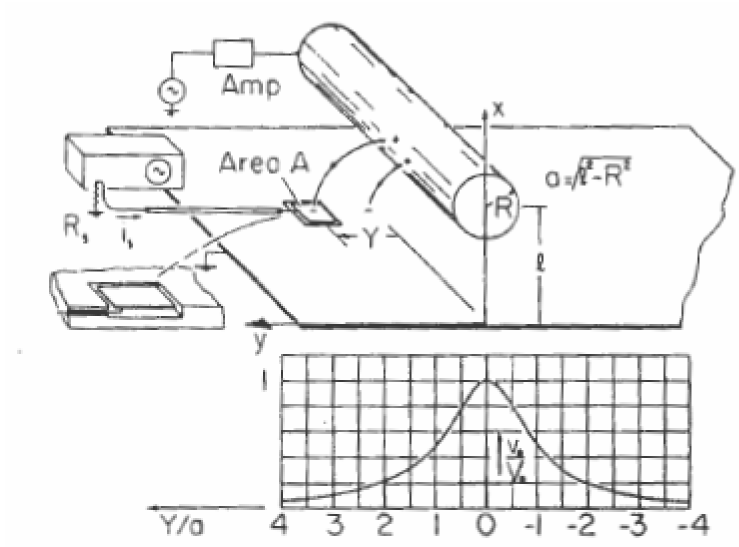
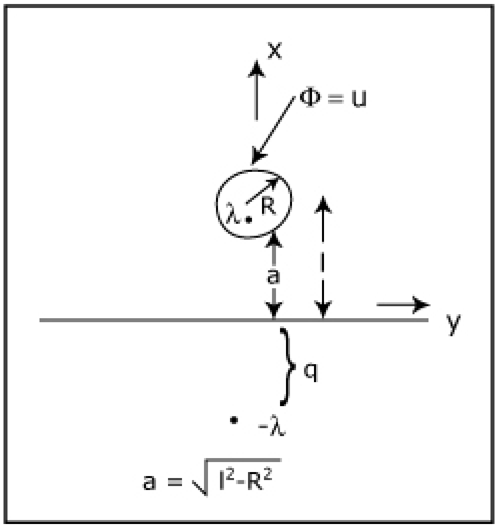


Figure 4.7.2 Charge induced on ground plane by overhead conductor is measured by probe. Distribution shown is predicted by (4.7.7).

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.



$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{\left[(a-x)^2+y^2\right]^{1/2}}{\left[(a+x)^2+y^2\right]^{1/2}} = \frac{-\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(a-x)^2+y^2}{(a+x)^2+y^2}\right]$$

$$C' = \frac{\lambda}{\Phi(x=l-R, y=0)} = \frac{\lambda}{\frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{a-l+R}{a+l-R}} = \frac{2\pi\varepsilon_0}{\ln\left[\frac{\sqrt{l^2-R^2}+l}{R}\right]}\ ,\ \ \Phi(x=l-R, y=0) = U$$

$$\sigma_s = \varepsilon_0 E_x(x=0) = -\varepsilon_0 \left.\frac{\partial\Phi}{\partial x}\right|_{x=0}$$

$$= \frac{+\not{\varepsilon_0}\lambda}{4\pi\not{\varepsilon_0}}\frac{d}{dx}\left[\ln\left[(a-x)^2+y^2\right]-\ln\left[(a+x)^2+y^2\right]\right]$$

$$= \frac{\lambda}{4\pi}\left.\left[\frac{-2(a-x)}{(a-x)^2+y^2}-\frac{2(a+x)}{(a+x)^2+y^2}\right]\right|_{x=0}$$

$$= \frac{-\lambda a}{\pi\left(a^2+y^2\right)}$$

Total Charge per unit length on ground plane is:

$$\lambda_T(x=0) = \int_{y=-\infty}^{\infty}\sigma_s dy = \int_{-\infty}^{\infty}\frac{-\lambda a}{\pi\left(a^2+y^2\right)}dy = \frac{-\lambda\not{a}}{\pi}\frac{1}{\not{a}}\left.\underbrace{\tan^{-1}\frac{y}{a}}_{\pi}\right|_{-\infty}^{\infty} = -\lambda$$

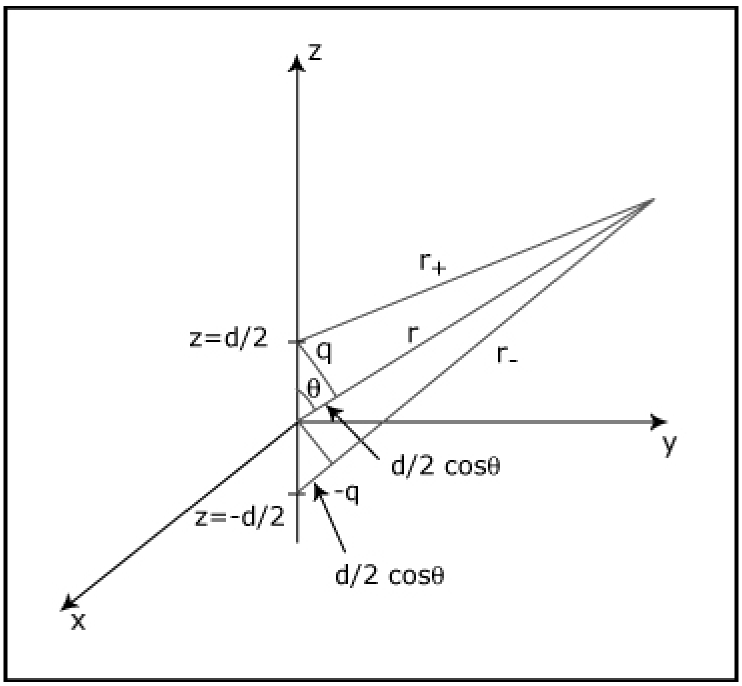
$$i_s = \frac{dq}{dt} \approx A\frac{d\sigma_s}{dt} = \frac{-aA}{\pi\left(a^2+y^2\right)}\frac{d\lambda}{dt} = \frac{-aAC'}{\pi\left(a^2+y^2\right)}\frac{dU}{dt}$$

take $U = U_0 \cos\omega t$

$$v_0 = -i_s R_s = -\frac{C'Aa}{\pi\left(a^2+y^2\right)} U_0 \omega \sin\omega t$$

V. Point Electric Dipole

1. Potential



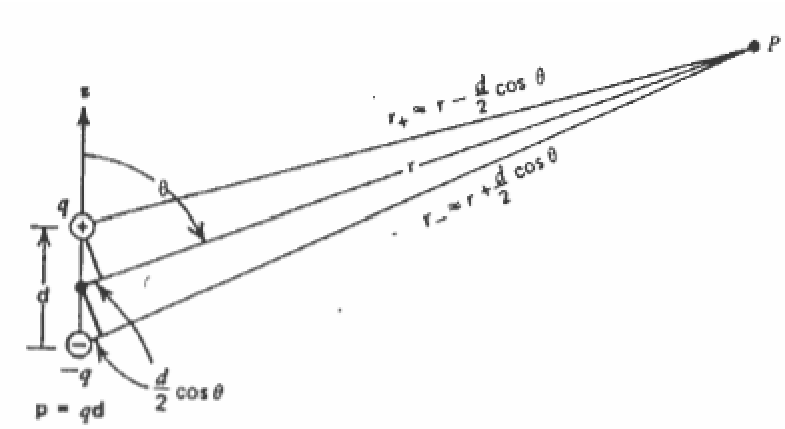
$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

$$r_+ = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$r_- = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

Note: $\Phi\left(z=0\right) = 0$

2. Point Electric Dipole (r>>d)



From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$r_+ \approx r - \frac{d}{2}\cos\theta \approx r\left[1 - \frac{d}{2r}\cos\theta\right]$$

$$r_- \approx r + \frac{d}{2}\cos\theta \approx r\left[1 + \frac{d}{2r}\cos\theta\right]$$

$$\Phi \approx \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{1 - \frac{d}{2r}\cos\theta} - \frac{1}{1 + \frac{d}{2r}\cos\theta}\right] \approx \frac{q}{4\pi\varepsilon_0 r}\left[1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right]$$

$$\approx \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\lim_{\substack{d\to 0\\ q\to\infty}} p = qd \ \text{(dipole moment)} \Rightarrow \Phi \approx \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\,\bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\bar{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\,\bar{i}_\phi\right]$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta\right]$$

3. Field Lines for Point Electric Dipole: $\frac{dr}{r d\theta} = \frac{E_r}{E_\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

$$\frac{dr}{r} = 2\cot\theta d\theta \Rightarrow \ln r = 2\ln(\sin\theta) + C$$

$$r = r_0 \sin^2\theta$$

$$r_0 = r\left(\theta = \pi/2\right)$$

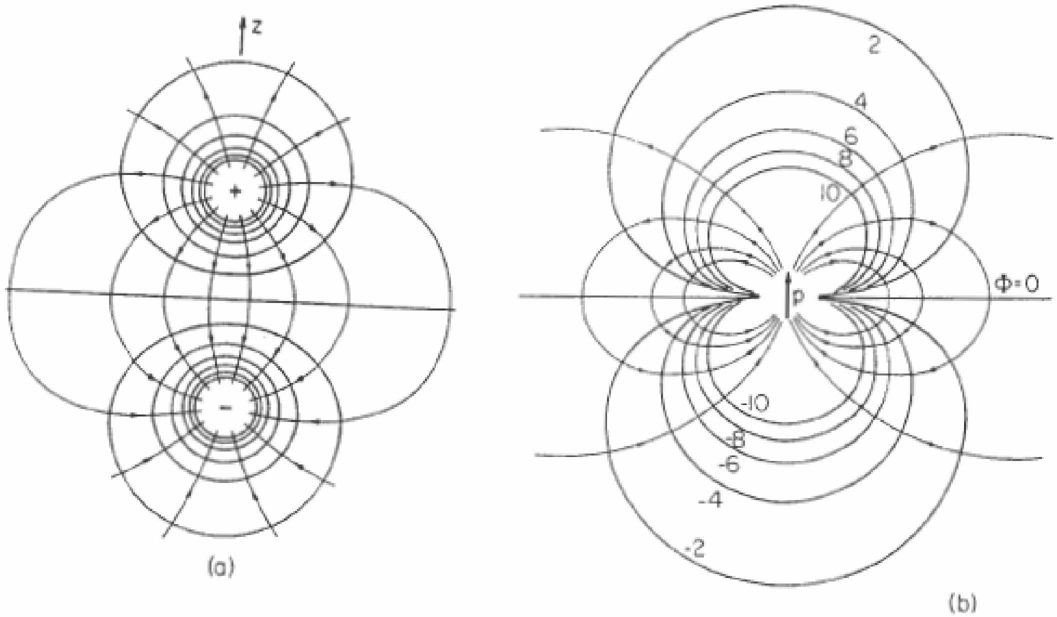


Figure 4.4.2 (a) Cross-section of equipotentials and lines of electric field intensity for the two charges of Figure 4.4.1. (b) Limit in which pair of charges form a dipole at the origin.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

VI. Line Current Above a Perfect Conductor

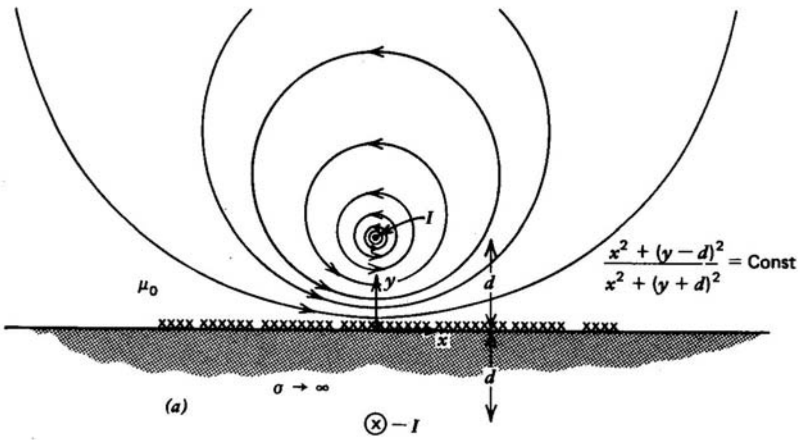


Figure 5-24 A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\bar{f}_I = \bar{I} \times \left(\mu_0 \bar{H}\right) \qquad \text{newtons / meter}$$

$$= I\,\bar{i}_z \times \left(\mu_0 \frac{I}{4\pi d}\,\bar{i}_x\right) = \frac{\mu_0 I^2}{4\pi d}\,\bar{i}_y$$