Notes on Linear Algebra for a Tutorial

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I. Introduction

We make use of some linear algebra concepts in 6.011 with the thought that most students have been exposed to relevant concepts previously, perhaps in high school, and perhaps in 18.03 or equivalent. However, it may be that not everyone has seen linear algebra before, and it may be that some students may be rusty. So, this provides motivation to develop some notes that cover some of the basic notions, hopefully in a simple way. In what follows we discuss some of the ideas, notation, and results that may be useful.

II. Linear equations in terms of a matrix and vectors

Perhaps the place to start is with a set of linear equations of the form

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 - x_2 &= 0 \end{aligned} \tag{1}$$

Probably we can solve these equations pretty quickly for x_1 and for x_2. However, given that linear equations come up often in mathematics, engineering and science, it would be good to have a systematic way of thinking about them. For example, we might consider these two equations as a particular example of a more general set of coupled equations of the form

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= b_1 \\ A_{21}x_1 + A_{22}x_2 &= b_2 \end{aligned} \tag{2}$$

There may be more complicated problems of this kind that involve more unknowns, such as

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3 \end{aligned} \tag{3}$$

Probably matrices were developed (long ago) originally to help with dealing with this kind of problem. For two linear equations we can make use of matrices and vectors to write

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{4}$$

where matrix multiplication of a vector is indicated in the intermediate step. A nice thing about this approach is that it allows us to write lots of coupled linear equations simply making use of the subscripts as

$$\sum_k A_{ij} x_j \ = \ b_j \tag{5}$$

In terms of matrix and vector notation this is written as

$$\mathbf{A}\mathbf{x} \ = \ \mathbf{b} \tag{6}$$

In the case of three unknowns we can write

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ = \ \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{bmatrix} \ = \ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{7}$$

This can be generalized to as many unknowns as we like. If we have the same number of equations as unknowns, then the matrix **A** will be a square matrix.

There are written records that this idea was known in antiquity; where a mathematical text written in the second century BCE in China discusses linear equations, solution by Gaussian elimination, and a treatment equivalent to the use of matrices and vectors as above.

III. Matrix multiplication

Once we have matrices, a natural question concerns what are the associated properties. This opens up the door to a potentially vast field of study, of which in these notes we will only touch on a few simple ones that we need for class. To proceed, we will need to know how to multiply matrices. We consider the multiplication of two matrices to make a third

$$\mathbf{A}\mathbf{B} \ = \ \mathbf{C} \tag{8}$$

in the case of square matrices. In the case of 2×2 matrices this can be written out as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \ = \ \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \ = \ \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{9}$$

We can also make use of the subscripts to write

$$C_{ij} \ = \ \sum_k A_{ik} B_{kj} \tag{10}$$

At this point it seems useful to consider the notion of an identity matrix, which in the 2×2 case can be written as

$$\mathbf{I} \ = \ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

We can evaluate the product

$$\mathbf{IA} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \;=\; \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \;=\; \mathbf{A} \tag{12}$$

Multiplication of a matrix by the identity matrix simply gives the matrix as a result. We can construct bigger (square) identity matrices similarly with 1 for diagonal elements and 0 for off-diagonal elements, which in general which satisfy

$$\mathbf{IA} \;=\; \mathbf{A} \tag{13}$$

The identity matrix multiplied by a vector gives the same vector back as well

$$\mathbf{Ix} \;=\; \mathbf{x} \tag{14}$$

IV. Eigenvectors and characteristic equations for eigenvalues

It is possible to find vectors which are able to produce scaled versions of themselves when multiplied by a matrix. For example, notice that

$$\begin{bmatrix} 1 & 1 \\ 10 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \begin{bmatrix} 3 \\ 6 \end{bmatrix} \;=\; 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{15}$$

This is a special case of

$$\mathbf{Av} \;=\; \lambda \mathbf{v} \tag{16}$$

where λ is a scalar and $\mathbf{v}$ is a vector.

We are interested in figuring out how to find the eigenvalues and eigenvectors. We might start with finding and equation for the eigenvalue λ in the simple case of a 2×2 matrix, in which case we can write

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;=\; \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{17}$$

This is equivalent to the linear equations

$$\begin{aligned} A_{11}v_1 + A_{12}v_2 &= \lambda v_1 \\ A_{21}v_1 + A_{22}v_2 &= \lambda v_2 \end{aligned} \tag{18}$$

Here the unknowns include v_1, v_2 and λ. It is possible to eliminate v_1 and v_2 to get an equation for λ. To do this we can first write

$$\begin{aligned} (A_{11} - \lambda)v_1 + A_{12}v_2 &= 0 \\ A_{21}v_1 + (A_{22} - \lambda)v_2 &= 0 \end{aligned} \tag{19}$$

and then eliminate v_1 by forming

$$
\begin{aligned}
A_{21}(A_{11}-\lambda)v_1 + A_{21}A_{12}v_2 &= 0 \\
(A_{11}-\lambda)A_{21}v_1 + (A_{11}-\lambda)(A_{22}-\lambda)v_2 &= 0
\end{aligned} \tag{20}
$$

and then subtracting to obtain

$$
\left[(A_{11}-\lambda)(A_{22}-\lambda) - A_{21}A_{12}\right]v_2 = 0 \tag{21}
$$

In the general case v_2 might not be zero, so we end up with a constraint on λ

$$
(A_{11}-\lambda)(A_{22}-\lambda) - A_{21}A_{12} = 0 \tag{22}
$$

This is the characteristic equation in the case of a 2×2 matrix that the eigenvalues must satisfy in order for the eigenvalue relation to be consistent.

We can repeat this kind of calculation in the case of a general 3×3 matrix, and get a more complicated characteristic equation. The result can be written as

$$
\begin{aligned}
&\lambda^3 - (A_{11}+A_{22}+A_{33})\lambda^2 + (A_{12}A_{21} + A_{13}A_{31} + A_{23}A_{32} - A_{11}A_{22} - A_{11}A_{33} - A_{22}A_{33})\lambda \\
&+A_{11}A_{22}A_{33} + A_{13}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{11}A_{23}A_{32} - A_{22}A_{13}A_{31} - A_{33}A_{12}A_{21} = 0
\end{aligned} \tag{23}
$$

There is no difficulty continuing this kind of calculation for bigger square matrices; however, it is clear that we will generate large numbers of terms. Instead of repeating this kind of elimination by hand each time, we would like to automate the calculation.

V. Characteristic equation in terms of a determinant

Today we recognize the development of the characteristic equation as deriving from the determinantal equation

$$
\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{24}
$$

The notion of the determinant originated in connection with the solution of sets of linear equations, where a solution to $\mathbf{Ax} = \mathbf{b}$ can be obtained if $\det(\mathbf{A}) \neq 0$. In 1683 Seki in Japan, and independently Liebnitz in Europe, evaluated the determinant for the lowest square matrices. There were earlier calculations for the 2×2 and 3×3 matrices that can retrospectively be interpreted as involving the evaluation of the determinant. It is likely that the Chinese mathematicians understood the need for the determinant to be nonzero to obtain a solution to a linear system of equations in antiquity.

We can think of the determinant as being defined as the result of the calculation above that eliminates the A_{ij} matrix elements in the eigenvalue calculation. This calculation has been well studied over the years, with the result that specific formulas were derived long ago for "small" matrices. You might have seen a more systematic approach that makes use of cofactors and minors (due to Laplace). Expansion formulas have been developed that describe the general case as well; one such formula can be written as

$$\det(\mathbf{A}) = \sum_{i_1} \cdots \sum_{i_n} \epsilon_{i_1 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n} \tag{25}$$

where $\epsilon_{i_1 \cdots i_n}$ is a Levi-Civita symbol which has a value of 0, +1 or -1 according to a test of the permutation of the indices.

For class this term we will be able to survive for the most part with determinantal formulas for the 2×2 case

$$\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A_{11}A_{22} - A_{12}A_{21} \tag{26}$$

and for the 3×3 case

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} =$$

$$A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \tag{27}$$

VI. Matrix inverse

If we have a single linear equation of the form

$$Ax = b \tag{28}$$

then if A is not zero we can solve it by writing

$$x = A^{-1}b \tag{29}$$

If we have a matrix and vector equation written as

$$\mathbf{Ax} = \mathbf{b} \tag{30}$$

then it seems to be reasonable that we might be able to write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{31}$$

For this to work we need to be able to compute the inverse of a matrix, and it would be good to know for what matrices this can work. For example, we know that in the scalar case above that if A is zero, then we are not going to be able to use the inverse. The same idea applies in the case of a matrix.

If we consider the case of a 2×2 matrix, we might start with

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{32}$$

We know that we can write these as linear equations of the form

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= b_1 \\ A_{21}x_1 + A_{22}x_2 &= b_2 \end{aligned} \tag{33}$$

We can solve these by multiplying by a factor and subtracting; for example, to eliminate x_1 we multiply by factors to get

$$\begin{aligned} A_{21}A_{11}x_1 + A_{21}A_{12}x_2 &= A_{21}b_1 \\ A_{11}A_{21}x_1 + A_{11}A_{22}x_2 &= A_{11}b_2 \end{aligned} \tag{34}$$

and then subtract to obtain

$$(A_{11}A_{22} - A_{12}A_{21})x_2 = A_{11}b_2 - A_{21}b_1 \tag{35}$$

If $A_{11}A_{22} - A_{12}A_{21}$ is not zero, then we can divide to obtain

$$x_2 = \frac{A_{11}b_2 - A_{21}b_1}{A_{11}A_{22} - A_{12}A_{21}} \tag{36}$$

For the other case we can write

$$\begin{aligned} A_{22}A_{11}x_1 + A_{22}A_{12}x_2 &= A_{22}b_1 \\ A_{12}A_{21}x_1 + A_{12}A_{22}x_2 &= A_{12}b_2 \end{aligned} \tag{37}$$

and subtract to obtain

$$(A_{22}A_{11} - A_{12}A_{21})x_1 = A_{22}b_1 - A_{12}b_2 \tag{38}$$

Once again if $A_{22}A_{11} - A_{12}A_{21}$ is not zero, then we can solve to obtain

$$x_1 = \frac{A_{22}b_1 - A_{12}b_2}{A_{22}A_{11} - A_{12}A_{21}} \tag{39}$$

These solutions can be combined to write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{A_{22}A_{11} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{40}$$

We recognize that for this approach to work we require

$$A_{22}A_{11} - A_{12}A_{21} \neq 0 \tag{41}$$

But we recognize that this can be written using the definition of the determinant for the 2×2 case as

$$\det(\mathbf{A}) \neq 0 \tag{42}$$

We will not be able to obtain a unique solution for a linear system if the determinant is zero, which is consistent with not being able to construct an inverse for $\mathbf{A}$ if the determinant is zero.

From the calculation above we conclude that the inverse for a 2×2 matrix is

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{43}$$

Although we will not need it this term, for the inverse of a 3×3 matrix, as long as the determinant is not zero, we can write

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} A_{22}A_{33} - A_{23}A_{32} & A_{13}A_{32} - A_{12}A_{33} & A_{12}A_{23} - A_{13}A_{22} \\ A_{23}A_{31} - A_{21}A_{33} & A_{11}A_{33} - A_{13}A_{31} & A_{13}A_{21} - A_{11}A_{23} \\ A_{21}A_{32} - A_{22}A_{31} & A_{12}A_{31} - A_{11}A_{32} & A_{11}A_{22} - A_{12}A_{21} \end{bmatrix} \tag{44}$$

For bigger square matrices it is possible to develop explicit formulas for the inverse as long as the determinant is not zero.

VII. Matrix of the eigenvectors

It is possible to construct a matrix for a 2×2 matrix $\mathbf{A}$ from the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ according to

$$\mathbf{V} = [\ \mathbf{v}_1\ \mathbf{v}_2\ ] \tag{45}$$

If we write the eigenvectors in terms of their elements

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 \qquad \mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 \tag{46}$$

then the $\mathbf{V}$ matrix might be written as

$$\mathbf{V} = \begin{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 & \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 \end{bmatrix} \tag{47}$$

To proceed we might adopt for this a notation of the form

$$\mathbf{V} = \begin{bmatrix} (v_1)_1 & (v_1)_2 \\ (v_2)_1 & (v_2)_2 \end{bmatrix} \tag{48}$$

where $(v_i)_k$ indicates the ith element of the vector of the kth eigenfunction.

Since this matrix is made up of the eigenvectors of $\mathbf{A}$, and since each of the eigenvectors satisfies the eigenvalue equation

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k \tag{49}$$

then we can write

$$\mathbf{AV} = \mathbf{A}[\ \mathbf{v}_1\ \mathbf{v}_2\ ] = [\ \lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2\ ] = \begin{bmatrix} \lambda_1(v_1)_1 & \lambda_2(v_1)_2 \\ \lambda_1(v_2)_1 & \lambda_2(v_2)_2 \end{bmatrix} \tag{50}$$

For the argument that follows, we will need to define a diagonal matrix made up of the eigenvalues according to

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{51}$$

If we multiply $\mathbf{V}$ and Λ, we get the same resulting matrix as we got when we multiplied $\mathbf{A}$ times $\mathbf{V}$

$$\mathbf{V\Lambda} = \begin{bmatrix} (v_1)_1 & (v_1)_2 \\ (v_2)_1 & (v_2)_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(v_1)_1 & \lambda_2(v_1)_2 \\ \lambda_1(v_2)_1 & \lambda_2(v_2)_2 \end{bmatrix} \tag{52}$$

Consequently, we can write

$$\mathbf{AV} = \mathbf{V\Lambda} \tag{53}$$

Although we found this to be true for the 2×2 case, it works for the other cases as well.

One reason that this is interesting is that it allows us to (formally) diagonalize the $\mathbf{A}$ matrix by writing

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{AV} \tag{54}$$

which works as long as the inverse matrix exists. In 6.011 we focus almost exclusively on matrices for which the eigenvalues are distinct, in which case the eigenvectors are independent, which is the conditions that the inverse matrix exists.

VIII. Application: Diagonalization of a state space model

One application of the matrix of eigenvectors is for the diagonalization of a state space model. Suppose that we start with a state space model of the form

$$\begin{aligned} \frac{d}{dt}\mathbf{q}(t) &= \mathbf{Aq}(t) + \mathbf{b}x(t) \\ y(t) &= \mathbf{c}^T\mathbf{q}(t) + dx(t) \end{aligned} \tag{55}$$

We would like to expand the state vector in terms of the eigenvectors according to

$$\mathbf{q}(t) = \mathbf{v}_1 r_1(t) + \mathbf{v}_2 r_2(t) \tag{56}$$

Based on the discussion above, we know that we can write this in terms of the matrix of eigenvectors according to

$$\mathbf{q}(t) = \mathbf{Vr}(t) \tag{57}$$

If we plug this into the evolution equation for the state vector $\mathbf{q}(t)$, we obtain

$$\frac{d}{dt}\mathbf{Vr}(t) = \mathbf{AVr}(t) + \mathbf{b}x(t) \tag{58}$$

Since the matrix of eigenvectors does not depend on time, this can be written as

$$\mathbf{V}\frac{d}{dt}\mathbf{r}(t) \;=\; \mathbf{AVr}(t) + \mathbf{b}x(t) \tag{59}$$

If the eigenvalues are independent, then the $\mathbf{V}$ matrix has an inverse, which leads to

$$\frac{d}{dt}\mathbf{r}(t) \;=\; \mathbf{V}^{-1}\mathbf{AVr}(t) + \mathbf{V}^{-1}\mathbf{b}x(t) \tag{60}$$

We recall that

$$\mathbf{V}^{-1}\mathbf{AV} \;=\; \mathbf{\Lambda} \tag{61}$$

If the $\mathbf{b}$ vector is written in terms of the eigenvectors

$$\mathbf{b} \;=\; \mathbf{v}_1\beta_1 + \mathbf{v}_2\beta_2 \;=\; \mathbf{V}\boldsymbol{\beta} \tag{62}$$

then we can express $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} \;=\; \mathbf{V}^{-1}\mathbf{b} \tag{63}$$

We end up with the state evolution equation written in terms of the mode amplitudes as

$$\frac{d}{dt}\mathbf{r}(t) \;=\; \mathbf{\Lambda r}(t) + \boldsymbol{\beta}x(t) \tag{64}$$

This is equivalent to

$$\begin{aligned}
\frac{d}{dt}r_1(t) \;&=\; \lambda_1 r_1(t) + \beta_1 x(t) \\
\frac{d}{dt}r_2(t) \;&=\; \lambda_2 r_2(t) + \beta_2 x(t)
\end{aligned} \tag{65}$$

There is no coupling between the different modes now, which makes it much easier to solve.

For the state space output equation we can write

$$y(t) \;=\; \mathbf{c}^T\mathbf{Vr}(t) + dx(t) \tag{66}$$

We can evaluate the product

$$\mathbf{c}^T\mathbf{V} \;=\; \mathbf{c}^T[\ \mathbf{v}_1\ \mathbf{v}_2\ ] \;=\; [\ \mathbf{c}^T\mathbf{v}_1\ \mathbf{c}^T\mathbf{v}_2\ ] \;=\; [\ \xi_1\ \xi_2\ ] \;=\; \boldsymbol{\xi}^T \tag{67}$$

We end up with

$$y(t) \;=\; \boldsymbol{\xi}^T\mathbf{r}(t) + dx(t) \tag{68}$$

which is equivalent to

$$y(t) \;=\; \xi_1 r_1(t) + \xi_2 r_2(t) + dx(t) \tag{69}$$

We now have the output in terms of the mode amplitudes.