What you need to know about bandlimited signals for 6.011

[1]A continuous-time (CT) signal $x_c(t)$ is termed *bandlimited* to ω_c rad/s or $f_c = \omega_c/(2\pi)$ Hz if its (CT) Fourier transform $X_c(j\omega)$, defined by

$$(1) \qquad\qquad X_c(j\omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\omega t}dt \ ,$$

is 0 for $|\omega| \geq \omega_c(< \infty)$ (we normally quote the smallest ω_c for which this is true). In other words, the signal has no frequency content at frequencies f_c Hz or higher, so it varies "smoothly" (or we could be more explicit and say "f_c-smoothly").

For such a signal we can write the inverse CTFT as:

$$(2) \qquad\qquad x_c(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} X_c(j\omega)e^{j\omega t}d\omega \ .$$

Note the limits on the integral!

Now consider the discrete-time (DT) signal $x_d[n]$ obtained by sampling $x_c(t)$ at intervals of T seconds. We label this sampling operation as continuous-to-discrete (C/D) transformation (it's the most common kind of C/D transformation, and the only kind we'll consider). We can now write

$$(3) \qquad\qquad x_d[n] = x_c(nT) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} X_c(j\omega)e^{j\omega nT}d\omega \ .$$

All that's happened here is we've written $t = nT$ in the exponent in Eq. (2). Making the change of variable $\Omega = \omega T$ rad, we can rewrite Eq. (3) as

$$(4) \qquad\qquad x_d[n] = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} \frac{X_c(j\omega)}{T}e^{j\Omega n}d\Omega \ ,$$

where $\Omega_c = \omega_c T$.

Compare the expression in Eq. (4) with the inverse DTFT formula that expresses the DT signal $x_d[n]$ in terms of its DTFT $X_d(e^{j\Omega})$:

$$(5) \qquad\qquad x_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(e^{j\Omega})e^{j\Omega n}d\Omega \ ,$$

where

$$(6) \qquad\qquad X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} \ .$$

We see from comparing Eqs. (4) and (5) that if $\Omega_c \leq \pi$, i.e., if $\omega_c T \leq \pi$, i.e., if the sampling frequency $f_s = 1/T$ satisfies

$$(7) \qquad\qquad f_s = \frac{1}{T} \geq \frac{\omega_c}{\pi} \geq 2f_c \ ,$$

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then (because the Fourier transform of a signal is unique)

$$(8) \qquad X_d(e^{j\Omega}) = \frac{X_c(j\omega)}{T}$$

for angular frequencies Ω in the interval $[-\pi, \pi]$, which we might refer to as the canonical or principal interval (and repeats periodically with period 2π outside that, as any DTFT must). This is Eq. (1.80) in the text.

Thus, if the sampling frequency is greater than twice the highest frequency present in the signal $x_c(t)$, we can reconstruct the CTFT of this CT signal, and hence the CT signal itself, from the DTFT of the sampled DT signal, and hence from the sampled signal itself. This is **Nyquist's sampling theorem**, and the relationship in Eq. (8) is key. This equation shows that for sampling at or above the Nyquist rate, we get the DTFT of the sampled sequence by simply scaling the CTFT amplitude by the factor $1/T$ and scaling the frequency axis by the factor T (to go from ω to $\Omega = \omega T$). And to go in the other direction, we simply perform the opposite scaling of amplitude and frequency axes.

Although many of the problems in Chapter 1 deal with cases where there is aliasing, i.e., where the sampling frequency does not satisfy the condition in Eq. (7), we shall not worry this semester about such refinements.

A small exercise (for you to do before reading further): Show that why Eq. (2) can be rewritten as

$$(9) \qquad x_c(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} T\Big(\sum_{n=-\infty}^{\infty} x_d[n] e^{-j\Omega n} \Big) e^{j\omega t} d\omega \ .$$

Assume now that T is set to the largest possible value that avoids aliasing, so $T = \pi/\omega_c$, i.e., we are sampling at the Nyquist rate. Interchanging summation and integration in the above expression, evaluating the integral on each term, and verifying that

$$(10) \qquad \int_{-\omega_c}^{\omega_c} e^{j\omega(t-nT)} d\omega = \frac{2\pi}{T} \frac{\sin\Big(\frac{\pi}{T}(t-nT) \Big)}{\frac{\pi}{T}(t-nT)} \ ,$$

we see that Eq. (9) yields a formula expressing the bandlimited signal $x_c(t)$ directly in terms of its samples:

$$(11) \qquad x_c(t) = \sum_{n=-\infty}^{\infty} x_d[n] \frac{\sin\Big(\frac{\pi}{T}(t-nT) \Big)}{\frac{\pi}{T}(t-nT)} \ .$$

To tease apart this expression, first look at the $n = 0$ term, which is $x_d[0] \sin(\pi t/T)/(\pi t/T)$. The unit-height sinc function $\sin(\pi t/T)/(\pi t/T)$ takes the value 1 at $t = 0$ and the value 0 at all other sampling instants, i.e., at all nonzero integer multiples of T, varying smoothly in between these points. Note also that the transform of this sinc function is constant at the value T for ω in $(-\omega_c, \omega_c)$, and is zero outside this; it has no frequencies higher than f_c Hz (and this is the sense in which it varies "smoothly"). Multiplying the unit-height sinc

by $x_d[0]$ scales it to a sinc function that takes the value $x_d[0]$ at $t = 0$ but is still 0 at all other sampling instants, and varies smoothly in between.

Now we do the same thing for the samples at all the other sampling instants: for a general sampling instant nT, we center a sinc function on that sampling time, which causes the time argument of the sinc to change from t to $t - nT$, and then scale this shifted sinc by $x_d[n]$. The combination of this infinite set of scaled and shifted sinc functions, each bandlimited to $(-\omega_c, \omega_c)$, is what creates the expression in Eq. (11), which we refer to as the *ideal bandlimited interpolation* of the samples $x_d[n]$, to create or reconstruct the bandlimited signal $x_c(t)$. The operation of generating a CT signal from a DT one in this fashion is referred to as *ideal* discrete-to-continuous (D/C) conversion.