Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Signals, Systems and Inference

FINAL EXAM

QUESTION & ANSWER BOOKLET

Your Full Name:	

The exam is **closed book**, but **4** sheets of notes (8 sides) are allowed. Calculators and other aids will not be necessary and are not allowed.

Check that this QUESTION & ANSWER BOOKLET has pages numbered up to 20. The booklet contains spaces for all relevant work and reasoning, and we'll hand out scratch paper as needed for your rough work. The prompt given above each answer space is an *abbreviated* version of the full question, so **read the full question, not just the prompt!**

Neat work and clear explanations count; show all relevant work and reasoning in the indicated spaces, because those spaces are all that we will be looking at in grading.

There are **5 problems**, for a total of 75 points. This roughly translates to your being able to spend about *2 minutes per point*, on average, so don't get too bogged down on a problem that is giving you inordinate trouble.

Problem	**Your Score**
1 (10 points)	
2 (19 points)	
3 (17 points)	
4 (10 points)	
5 (19 points)	
Total (75 points)	

Problem 1 (10 points)

Suppose we are given a real and finite-energy (but otherwise arbitrary) DT signal $w[n]$, with associated DTFT $W(e^{j\Omega})$. We want to approximate $w[n]$ by another real, finite-energy DT signal $y[n]$ that is bandlimited to the frequency range $|\Omega| < \pi/4$ (within the usual $[-\pi, \pi]$ interval for Ω); so $Y(e^{j\Omega})$ is zero for $|\Omega| \geq \pi/4$. Apart from this constraint on its bandwidth, we are free to choose $y[n]$ as needed to get the best approximation.

Suppose we measure the quality of approximation by the following sum-of-squared-errors criterion:

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} (w[n] - y[n])^2 \ .$$

Our problem is then to minimize $\mathcal{E}$ by appropriate choice of the bandlimited $y[n]$, given the signal $w[n]$. This problem leads you through to the solution.

(a) (3 points) Express $\mathcal{E}$ in terms of a frequency-domain integral on the interval $|\Omega| \leq \pi$ that involves $W(e^{j\Omega}) - Y(e^{j\Omega})$.

(b) (5 points) Write your integral from (a) as a sum of integrals, one over each of the following ranges: $-\pi \leq \Omega \leq -\pi/4$, $-\pi/4 < \Omega < \pi/4$, and $\pi/4 \leq \Omega \leq \pi$. Use this to deduce how $Y(e^{j\Omega})$ needs to be picked in order to minimize $\mathcal{E}$, and what the resulting minimum value of $\mathcal{E}$ is. (*Hint:* Resist the temptation in this case to expand out $|a - b|^2$, for complex a and b, as $|a|^2 + |b|^2 - ab^* - a^*b$.)

(c) (2 points) Using your result in (b), write down an explicit formula for the $y[n]$ that minimizes $\mathcal{E}$, expressing this $y[n]$ as a suitable integral involving $W(e^{j\Omega})$.

1(a) (3 points) With $\mathcal{E} = \sum_{n=-\infty}^{\infty} (w[n] - y[n])^2$, express $\mathcal{E}$ in terms of a frequency-domain integral on the interval $|\Omega| \leq \pi$ that involves $W(e^{j\Omega}) - Y(e^{j\Omega})$.

$\mathcal{E} =$

1(b) (5 points) Write your integral from 1(a) as a sum of integrals, one over each of the following ranges: $-\pi \leq \Omega < -\pi/4$, $-\pi/4 \leq \Omega \leq \pi/4$, and $\pi/4 < \Omega \leq \pi$:

Use this to deduce how $Y(e^{j\Omega})$ needs to be picked in order to minimize $\mathcal{E}$ (resist the temptation in this problem to expand out $|a-b|^2$, for complex a and b, as $|a|^2 + |b|^2 - ab^* - a^*b$):

$Y(e^{j\Omega}) =$

Resulting minimum value of $\mathcal{E} =$

1(c) (2 points) Using your result in 1(b), write down an explicit formula for the $y[n]$ that minimizes $\mathcal{E}$, expressing this $y[n]$ as a suitable integral involving $W(e^{j\Omega})$:

$y[n] =$

Problem 2 (19 points)

A continuous-time state-space model for the spread of fashion through a community takes the form

$$\begin{aligned}\frac{d}{dt}q_1(t) &= -\gamma\, q_1(t)q_2(t) + x(t) \\ \frac{d}{dt}q_2(t) &= \gamma\, q_1(t)q_2(t) - \epsilon\, q_2(t) \\ y(t) &= \eta\, q_2(t)\ .\end{aligned}$$

The state variable $q_1(t)$ denotes the number of people (approximated as a real number) susceptible to adopting the fashion at time t, and $q_2(t)$ denotes the number (again a real number) that have adopted the fashion at this time. In this model, the rate at which the fashion is adopted is proportional (with proportionality constant γ) to the product of the susceptible and fashionable populations. The constant ϵ reflects the rate at which fashionable people outgrow/abandon the fashion. The quantity $x(t)$ represents a control input that denotes the rate at which susceptibles are replenished (perhaps through advertising efforts) and $y(t)$ is a measured output, proportional (with proportionality constant η) to the number of fashionables.

(a) (2 points) Suppose $x(t)$ is held constant at the positive value $\overline{x} > 0$. Show that there is precisely one equilibrium point, and determine the corresponding equilibrium values $\overline{q}_1$ and $\overline{q}_2$ of the state variables, expressed in terms of the problem parameters.

(b) (8 points) Write down the linearized model at this equilibrium point. This model approximately governs the deviations $\widetilde{q}_1(t) = q_1(t) - \overline{q}_1$, $\widetilde{q}_2(t) = q_2(t) - \overline{q}_2$, $\widetilde{x}(t) = x(t) - \overline{x}$ and $\widetilde{y}(t) = y(t) - \overline{y}$ when these are sufficiently small.

As a check, for some choice of parameters in the original nonlinear model, your linearized model should take the form

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\widetilde{q}_1(t)\\ \widetilde{q}_2(t)\end{bmatrix} &= \begin{bmatrix}-3 & -1\\ 3 & 0\end{bmatrix}\begin{bmatrix}\widetilde{q}_1(t)\\ \widetilde{q}_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\widetilde{x}(t) \\ \widetilde{y}(t) &= \begin{bmatrix}0 & 0.1\end{bmatrix}\begin{bmatrix}\widetilde{q}_1(t)\\ \widetilde{q}_2(t)\end{bmatrix}\ .\end{aligned}$$

What choice of $\gamma\overline{x}$, ϵ and η yields the above numerical values? And which – if any – of the three 0 entries that are shown in this linearized model occur only for a particular combination of parameters?

Use the linearized model with these specific numerical entries for the rest of this problem.

(c) (4 points) Determine if the linearized system in (b) is: (i) reachable; (ii) observable. You will get **2 points extra credit** if you do this **without** explicitly computing eigenvectors. Show all relevant reasoning and computations.

(d) (5 points) Suppose we implement an **output feedback** of the form $\widetilde{x}(t) = g\,\widetilde{y}(t)$. Write down the resulting (linearized) closed-loop system in the form

$$\frac{d}{dt}\widetilde{\mathbf{q}}(t) = \mathbf{A}_c\,\widetilde{\mathbf{q}}(t)\ ,$$

and determine what choice of output feedback gain g, **if any**, will result in the eigenvalues of $\mathbf{A}_c$ being -1 and -2.

$$\begin{aligned}
\frac{d}{dt}q_1(t) &= -\gamma\, q_1(t)q_2(t) + x(t) \\
\frac{d}{dt}q_2(t) &= \gamma\, q_1(t)q_2(t) - \epsilon\, q_2(t) \\
y(t) &= \eta\, q_2(t)\ .
\end{aligned}$$

2(a) (2 points) Suppose $x(t) = \overline{x} > 0$. Show that there is precisely one equilibrium point, and determine the equilibrium values $\overline{q}_1$ and $\overline{q}_2$ of the state variables:

$\overline{q}_1 =$, $\overline{q}_2 =$

2(b) (8 points) Write down the linearized model at this equilibrium point. This model approximately governs the deviations $\widetilde{q}_1(t) = q_1(t) - \overline{q}_1$, $\widetilde{q}_2(t) = q_2(t) - \overline{q}_2$, $\widetilde{x}(t) = x(t) - \overline{x}$ and $\widetilde{y}(t) = y(t) - \overline{y}$ when these are sufficiently small.

As a check, for some choice of parameters in the original nonlinear model, your linearized model should take the form

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{bmatrix} &= \begin{bmatrix} -3 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\widetilde{x}(t) \\ \widetilde{y}(t) &= \begin{bmatrix} 0 & 0.1 \end{bmatrix}\begin{bmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{bmatrix} . \end{aligned}$$

What choice of $\gamma\overline{x}$, ϵ, and η yields the above numerical values?

$\gamma\overline{x} =$, $\epsilon =$ and $\eta =$

And which – if any – of the three 0 entries that are shown in this linearized model occur only for a particular combination of parameters?

2(c) (4 points) Determine if the linearized system in 2(b) is: (i) reachable; (ii) observable. You will get **2 points extra credit** if you do this **without** explicitly computing eigenvectors. Show all relevant reasoning and computations.

(i) Reachability:

(ii) Observability:

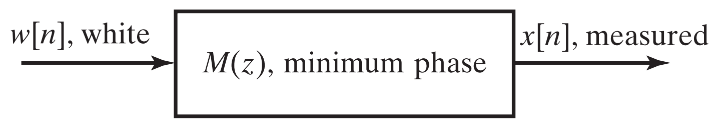
2(d) (5 points) Suppose we implement an **output feedback** of the form $\widetilde{x}(t) = g\,\widetilde{y}(t)$. Write down the resulting (linearized) closed-loop system in the form

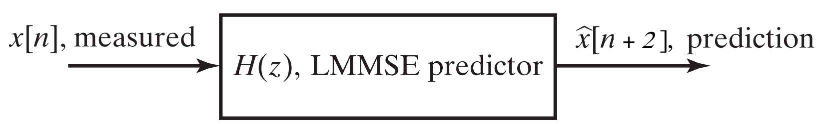
$$\frac{d}{dt}\widetilde{\mathbf{q}}(t) = \mathbf{A}_c\,\widetilde{\mathbf{q}}(t)\ ,$$

and determine what choice of output feedback gain g, **if any**, will result in the eigenvalues of $\mathbf{A}_c$ being -1 and -2.

$\mathbf{A}_c =$

$g =$





Problem 3 (17 points)

We have measurements of a WSS random process $x[n]$ that is modeled as the output of a minimum-phase LTI system whose input is a white process $w[n]$, with $E\{w^2[n]\} = 1$. (Recall that a minimum-phase DT system is defined as *stable, causal, and with a stable, causal inverse.*) The situation is shown in the upper figure.

Suppose the transfer function of the system above is

$$M(z) = \frac{\gamma}{z-\lambda} + d = d\,\frac{z - (\lambda - \frac{\gamma}{d})}{z-\lambda}\ ,$$

where $\gamma \neq 0$ and $d \neq 0$. You may also find it helpful to note that

$$M(z) = d + \gamma z^{-1}\Big(1 + \lambda z^{-1} + \lambda^2 z^{-2} + \cdots\Big)\ .$$

(a) (2 points) What is the expected value of $x[n]$? (No points unless you explain your reasoning fully!)

(b) (2 points) What is the fluctuation spectral density $D_{xx}(e^{j\Omega})$ of $x[\cdot]$, expressed in terms of the given quantities γ, λ, d? (It suffices to have a correct expression; you need not simplify your expression.)

(c) (3 points) Suppose $y[\cdot]$ is some other process that is jointly wide-sense stationary with $x[\cdot]$ (and hence with $w[\cdot]$ too, though we don't ask you to explain why). Express $D_{yw}(z)$ in terms of $D_{yx}(z)$ and $M(z)$ (and/or closely related quantities, if you think these are needed).

We would like to pass the process $x[n]$ through a stable LTI filter with system function $H(z)$ that is chosen to make this filter the LMMSE estimator of $x[n+2]$, i.e., the LMMSE **two-step** predictor, as shown in the lower figure on the preceding page. Denote the resulting estimate by $\widehat{x}\,[n+2]$.

(d) (3 points) Suppose there are no constraints on the LTI filter $H(z)$ beyond stability. Determine the optimum $H(z)$ and draw a fully labeled sketch of the associated unit sample response $h[n]$. Also determine the associated MMSE,

$$E\{(x[n+2] - \widehat{x}\,[n+2])^2\} \ .$$

(e) (6 points) Suppose now that we constrain the filter $H(z)$ to not only be stable but also **causal**. Again determine the optimum filter and the associated mean square error (explaining your reasoning!). Your answers will be expressed in terms of the given parameters, namely γ, λ, and d.

(f) (1 point) Returning to the unit sample response of the optimal unconstrained filter in (d), suppose you were to set all the negative-time values of that $h[n]$ to 0, would you get the unit sample response of the optimal causal filter in (e)?

3(a) (2 points) What is the expected value of $x[n]$? (No points unless you explain your reasoning fully!)

$E[x[n]] =$

3(b) (2 points) What is the fluctuation spectral density $D_{xx}(e^{j\Omega})$ of $x[\cdot]$, expressed in terms of the given quantities γ, λ, d? (It suffices to have a correct expression; you need not simplify your expression.)

$D_{xx}(e^{j\Omega}) =$

3(c) (3 points) Suppose $y[\cdot]$ is some other process that is jointly wide-sense stationary with $x[\cdot]$ (and hence with $w[\cdot]$ too, though we don't ask you to explain why). Express $D_{yw}(z)$ in terms of $D_{yx}(z)$ and $M(z)$ (and/or closely related quantities, if you think these are needed).

$D_{yw}(z) =$

3(d) (3 points) Suppose there are no constraints on the LTI filter $H(z)$ beyond stability. Determine the optimum $H(z)$ and draw a fully labeled sketch of the associated unit sample response $h[n]$. Also determine the associated MMSE, $E\{(x[n+2] - \widehat{x}\,[n+2])^2\}$.

$H(z) =$

Associated MMSE =

Plot of $h[n]$:

3(e) Suppose now that we constrain the filter $H(z)$ to not only be stable but also **causal**. Again determine the optimum filter and the associated mean square error (explaining your reasoning!). Your answers will be expressed in terms of the given parameters, namely γ, λ, and d. (**Start work here, continue on next page.**)

3(e) continued (6 points)

$H(z) =$

Associated MMSE =

3(f) (1 point) Returning to the unit sample response of the optimal unconstrained filter in 3(d), suppose you were to set all the negative-time values of that $h[n]$ to 0, would you get the unit sample response of the optimal causal filter in 3(e)?

Problem 4 (10 points)

Suppose that

$$X = S + W$$

where S and W are independent Gaussian random variables with respective means μ_S, μ_W and respective variances σ_S^2, σ_W^2.

(a) (3 points) Is X guaranteed to be a Gaussian random variable? (Be sure to state the reasoning behind your answer, otherwise you will lose points.) Also write down the mean and variance of X.

(b) (6 points) Let $\widehat{s}(X)$ denote the LMMSE estimator of S from measurement of X. Obtain an expression for this estimator, and for its associated mean square error, expressed in terms of the given parameters.

(c) (1 point) Can the MMSE estimator do better in this case? (To get the point for this part, you will need to explain your answer.)

4(a) (3 points) Is X guaranteed to be a Gaussian random variable? (Be sure to state the reasoning behind your answer, otherwise you will lose points.) Also write down the mean and variance of X.

Is X guaranteed to be Gaussian?

$\mu_X =$, $\sigma_X^2 =$

4(b) (6 points) Let $\widehat{s}(X)$ denote the LMMSE estimator of S from measurement of X. Obtain an expression for this estimator, and for its associated mean square error, expressed in terms of the given parameters.

$\widehat{s}(X) =$

MMSE $=$

4(c) (1 point) Can the MMSE estimator do better in this case? (To get the point for this part, you will need to explain your answer.)

Problem 5 (19 points)

A signal $X[n]$ that we will be measuring for $n = 1, 2, \ldots, L$ is known to be generated according to one of the following two hypotheses:

$$\begin{aligned} H_0: \quad X[n] &= W[n] \quad \text{holds with } \textit{a priori} \text{ probability } P(H_0) = p_0 \;, \\ H_1: \quad X[n] &= V[n] \quad \text{holds with } \textit{a priori} \text{ probability } P(H_1) = p_1 = 1 - p_0 \;. \end{aligned}$$

Here $W[n]$ is a zero-mean i.i.d. Gaussian process with known constant variance σ_W^2 at each time instant, i.e., the value at each time instant is governed by the probability density function

$$f_W(w) = \frac{1}{\sigma_W \sqrt{2\pi}} \exp\Big\{ -\frac{w^2}{2\sigma_W^2} \Big\}$$

and the values at different times are independent of each other. Similarly, $V[n]$ is a zero-mean Gaussian process, taking values that are independent at distinct times, but with a variance that changes in a known manner over time, so the variance at time n is known to be σ_n^2. We will find it notationally helpful in working through this problem to use the definition

$$\xi[n] = \Big(\frac{1}{\sigma_W^2} - \frac{1}{\sigma_n^2} \Big) \;.$$

Note that $\xi[n]$ may be positive for some n but negative or zero for others, corresponding to having $\sigma_W < \sigma_n$, $\sigma_W > \sigma_n$ or $\sigma_W = \sigma_n$ respectively.

(a) (5 points) Suppose we only have a measurement at $n = 1$, with $X[1] = x[1]$. Show that the decision rule for choosing between H_0 and H_1 with minimum probability of error, given this measurement, takes the form

$$\xi[1] \Big(x[1] \Big)^2 \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} \gamma$$

for some appropriately chosen threshold γ. Specify this γ in terms of the problem parameters.

(b) (5 points) With your result from (a), but now assuming $\xi[1] > 0$, sketch and label the two conditional densities—namely $f_{X[1]|H}(x|H_0)$ and $f_{X[1]|H}(x|H_1)$—that govern $X[1]$ under the two respective hypotheses.

Assuming that the two hypotheses are equally likely so $p_0 = p_1$, mark in the points $\pm\sqrt{\gamma/\xi[1]}$ on the horizontal (i.e., x) axis, then shade in the region or regions whose total area yields the conditional probability $P(\text{`}H_1\text{'}|H_0)$, and express this conditional probability in terms of the standard Q function,

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\nu^2/2}\, d\nu \ .$$

(c) (3 points) With the same situation as in (b), but with the hypotheses no longer restricted to be equally likely *a priori*, specify the range of values for p_0 in which the optimal decision will always be 'H_1', no matter what the measured value $x[1]$.

(d) (6 points) Now suppose we have measurements at $n = 1, 2, \ldots, L$, i.e., we know $X[1] = x[1], X[2] = x[2], \ldots, X[L] = x[L]$. Determine the decision rule for minimum probability of error, writing it in a form that generalizes your result from (a).

5(a) Suppose we only have a measurement at $n = 1$, with $X[1] = x[1]$. Show that the decision rule for choosing between H_0 and H_1 with minimum probability of error, given this measurement, takes the form

$$\xi[1]\Big(x[1]\Big)^2 \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} \gamma\ , \qquad \xi[1] = \Big(\frac{1}{\sigma_W^2} - \frac{1}{\sigma_1^2}\Big)\ .$$

for some appropriately chosen threshold γ. Specify this γ in terms of the problem parameters. (**Start work here and continue on next page.**)

5(a) continued (5 points)

$\gamma =$

5(b) (5 points) Assuming $\xi[1] > 0$, sketch and label $f_{X[1]|H}(x|H_0)$ and $f_{X[1]|H}(x|H_1)$. Assuming $p_0 = p_1$, carefully mark in the points $\pm\sqrt{\gamma/\xi[1]}$ on the horizontal (i.e., x) axis, then shade in the region or regions whose total area yields the conditional probability $P(\text{'}H_1\text{'}|H_0)$, and express this conditional probability in terms of the standard Q function, $Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\nu^2/2}\, d\nu$.

$P(\text{'}H_1\text{'}|H_0) =$

5(c) (3 points) With the same situation as in 5(b), but with the hypotheses no longer restricted to be equally likely *a priori*, specify the range of values for p_0 in which the optimal decision will always be 'H_1', no matter what the measured value $x[1]$.

Range of p_0 for which we will always pick 'H_1':

5(d) Now suppose we have measurements at $n = 1, 2, \ldots, L$, i.e., we know $X[1] = x[1], X[2] = x[2], \ldots, X[L] = x[L]$. Determine the decision rule for minimum probability of error, writing it in a form that generalizes your result from 5(a). (**Start your work here, continue on next page.**)

5(d) continued (6 points)

Decision rule:

[**Optional reading**: For the special case in which $V[n] = S[n] + W[n]$, where $S[n]$ is a zero-mean i.i.d. Gaussian process that is independent of $W[\cdot]$ and has variance $\sigma_S^2[n]$, the decision rule from 5(d) can be written as a comparison of the quantity

$$\sum_{n=1}^{L} x[n]\, \widehat{s}_n(x[n]) \tag{1}$$

with a fixed threshold, where $\widehat{s}_n(X[n])$ denotes the LMMSE estimator of $S[n]$ from measurement of $X[n]$ under hypothesis H_1; this is the estimator you derived in Problem 4(b). This form of the decision rule is similar to what we obtained in the case of a deterministic signal.]

Thanks for taking 6.011. Enjoy the summer, and if you're graduating and leaving, all good wishes for your life beyond MIT!